7 th Edition

As per New Syllabus 2019
UNIVERSITY OF MUMBAI

ENGINEERING PHYSICS-II

F. E. Semester-II

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P. Jamnadas LLP.

PREFACE

"The Science of today is the Technology of tomorrow." .... Edward Teller

It is said that Science is the mother of all Technologies. Though technologies change very rapidly the basic concepts of Science remain largely unchanged. Physics, a major section of Science plays a pivotal role in the field of Engineering. The subject, Engineering Physics bridges the gap between theoretical Physics and Applied Engineering.

This book on Engineering Physics is written according to the revised (2019-2020) syllabus of The University of Mumbai. The objective of this course is to understand the basic concepts of Physics and the founding principles of technology. This course will develop the scientific temperament in the learners for scientific observations, recording and inference drawing essentials for technological studies.

The salient features of the book are :

i) The subject is presented in a very simple and lucid manner.

ii) The topics are explained in a logical and systematic way with plenty of illustrative simple diagrams.

iii) The mathematical parts are treated very methodically with related solved examples.

iv) Modules are complete with previous university question papers along with the solutions.

v) Above all to make this book a textbook plenty of systematically illustrated Mathematical Problems and Review Questions are retained and expanded.

I hope that this book will serve as an important learning resource for the students of Engineering courses.

Feedback and suggestions from the readers will be really appreciated.

Dr. Swati Bawra

SYLLABUS

Engineering Physics - II

First Year Engineering (Semester - II)

(Mumbai University - Common for All Branches of Engineering)

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1. Diffraction **(04 Hours)**

(**Prerequisites :** Wave front and Huygen's principle, Reflection and refraction, Diffraction, Fresnel diffraction and Fraunhoffer diffraction.)

Diffraction : Fraunhoffer diffraction at single slit, Diffraction Grating, Resolving power of a grating, Applications of diffraction grating, Determination of wavelength of light using plane transmission grating.

2. Laser and Fibre Optics **(06 Hours)**

(**Prerequisites :** Absorption, recombination. energy bands of p-n junction, Refractive index of a material, Snell's law.)

Laser : Spontaneous emission and stimulated emission, Metastable state, Population inversion, Types of pumping, Resonant cavity, Einsteins's equations, Helium Neon laser, Nd:YAG laser, Semiconductor laser, Applications of laser- Holography.

Fibre optics : Numerical Aperture for step index fibre, Critical angle, Angle of acceptance, V number, Number of modes of propagation, Types of optical fibres, Fibre optic communication system.

3. Electrodynamics **(03 Hours)**

(**Prerequisites :** Electric Charges, Coulomb's law-force between two point charges, Electric field, Electric field due to a point charge, Electric field lines, Electric dipole, Electric field due to a dipole, Gauss's law, Faraday's law.)

Scalar and Vector field, Physical significance of gradient, Curl and divergence in Cartesian co-ordinate system, Gauss's law for electrostatics, Gauss's law for magnetostatics, Faraday's Law and Ampere's circuital law, Maxwell's equations (Free space and time varying fields).

4. Relativity **(02 Hours)**

(**Prerequisites :** Cartesian co-ordinate system)

Special theory of Relativity : Inertial and Non-inertial Frames of reference, Galilean transformations, Lorentz transformations (Space-time coordinates), Time Dilation, Length Contraction and Mass-Energy relation.

5. Nanotechnology **(04 Hours)**

(**Prerequisites :** Scattering of electrons, Tunneling effect, Electrostatic focusing, Magneto static focusing.)

Nanomaterials : Properties (Optical, electrical, magnetic, structural, mechanical) applications, Surface to volume ratio, Two main approaches in nanotechnology - Botto technique and Top down technique.

Tools for characterization of Nanoparticles : Scanning Electron Microscope (S Transmission Electron Microscope (TEM), Atomic Force Microscope (AFM). Metho synthesize Nanomaterials : Ball milling, Sputtering, Vapour deposition, Solgel.

6. Physics of Sensors **(05 H**

(Prerequisites : Transducer concept, Meaning of calibration, Piezoelectric effect.)

Resistive Sensors :

(a) Temperature measurement : Pt100 construction, calibration,

(b) Humidity measurement using resistive sensors.

Pressure Sensor : Concept of pressure sensing by capacitive, Flux and inductive me Analog pressure sensor — Construction, Working and Calibration and Applications.

Piezoelectric Transducers : Concept of piezoelectricity, Use of piezoelectric trans as ultrasonic generator. Application of ultrasonic transducer for distance measure Liquid and air velocity measurement.

Optical sensor : Photodiode, Construction and use of photodiode as ambient measurement and flux measurement.

Pyroelectric Sensors : Construction and working principle, Application of pyroelectric se as bolometer.

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MODULE 1

Diffraction

(**Prerequisites :** Wave front and Huygen's principle, Reflection and refraction, Diffraction, Fresnel diffraction and Fraunhoffer diffraction.)

Diffraction : Fraunhoffer diffraction at single slit, Diffraction Grating, Resolving power of a grating, Applications of diffraction grating, Determination of wavelength of light using plane transmission grating . **(04 Hours)**

(Weightage - 15%)

Course Outcome : CO1 : Learner will be able to illustrate the knowledge of diffraction through slits and its applications.

SYNOPSIS

1.1 Introduction

1.2 Prerequisites

1.3 Fraunhoffer Diffraction

1.4 Diffraction Grating and its Characteristics

1.5 Resolving Power of a Diffraction Grating

1.6 Application of Diffraction Grating : Determination of Unknown Wavelength of Light by Diffraction Grating

1.7 Solved Problems

Important Points to Remember

Exercise

Previous University Examination Questions with Solutions

1.1 Introduction

It is a matter of common experience that waves bend round obstacles placed in ... path. The amount of bending, however, depends upon the size of the obstacle and wavelength of the incident wave. Thus when an obstacle or a small aperture, of the o... of the wavelength of light, is placed in the path of light, the light deviates from straight propagation and enters in the geometric shadow. This phenomenon is called diffractio... light which is an important characteristic of wave motion.

Examples of diffraction effect in day to day life are :

(i) A series of dark lines parallel to the fingers when one tries to view a st... source of light through the gap between two closely spaced fingers.

(ii) The colours seen on a compact disc.

The phenomenon of diffraction leads to a basic limitation in resolution of instruments like camera, telescope, microscope, etc.

The diffraction effect is pronounced if the size of the obstacle or aperture is of order of the wavelength of the incident waves. As the wavelength of visible l... ($\sim 10^{-6}$ m) is much smaller than the size of the objects around us diffraction effect is much visible in our day to day life.

1.2 Prerequisites

1.2.1 : Wave Front and Huygen's Principle

- A wave is a disturbance that propagates through a medium in space and time. Duri... a wave motion the particles of the medium oscillate about their mean position as energy is transferred from one particle to another. Every particle begins to vibra... little later than its predecessor. Hence, there is a progressive change of phase fr... particle to particle in the direction of wave propagation.
- Huygen introduced **the concept of a wave front which is an imaginary surf... formed by the equiphase points of a wave motion.** For a sinusoidal plane w... the wave fronts are planes perpendicular to the direction of propagation.
- During a wave motion, the propagation of a wave front is explained by Huyge... principle as follows :

(i) Each point on the given wave front acts as a source of secondary wavelets

(ii) The secondary wavelets from each point travel through space in all directions with velocity of light.

(iii) A surface touching the secondary wavelets tangentially in the forward direction at any given time constructs the new wave front at that instant. This is known as **secondary wave front.**

The propagation of a wave is visualized as the propagation of the wave fronts.

For a point source of light the emitted energy propagates uniformly in all directions and the wave fronts are spherical as seen in Fig. 1.1 (a). These waves are called spherical waves.

For broad source that produces a parallel beam of light the energy propagates uniformly in one direction and the wave fronts are planes as seen in Fig. 1.1 (b). These waves are called plane waves.

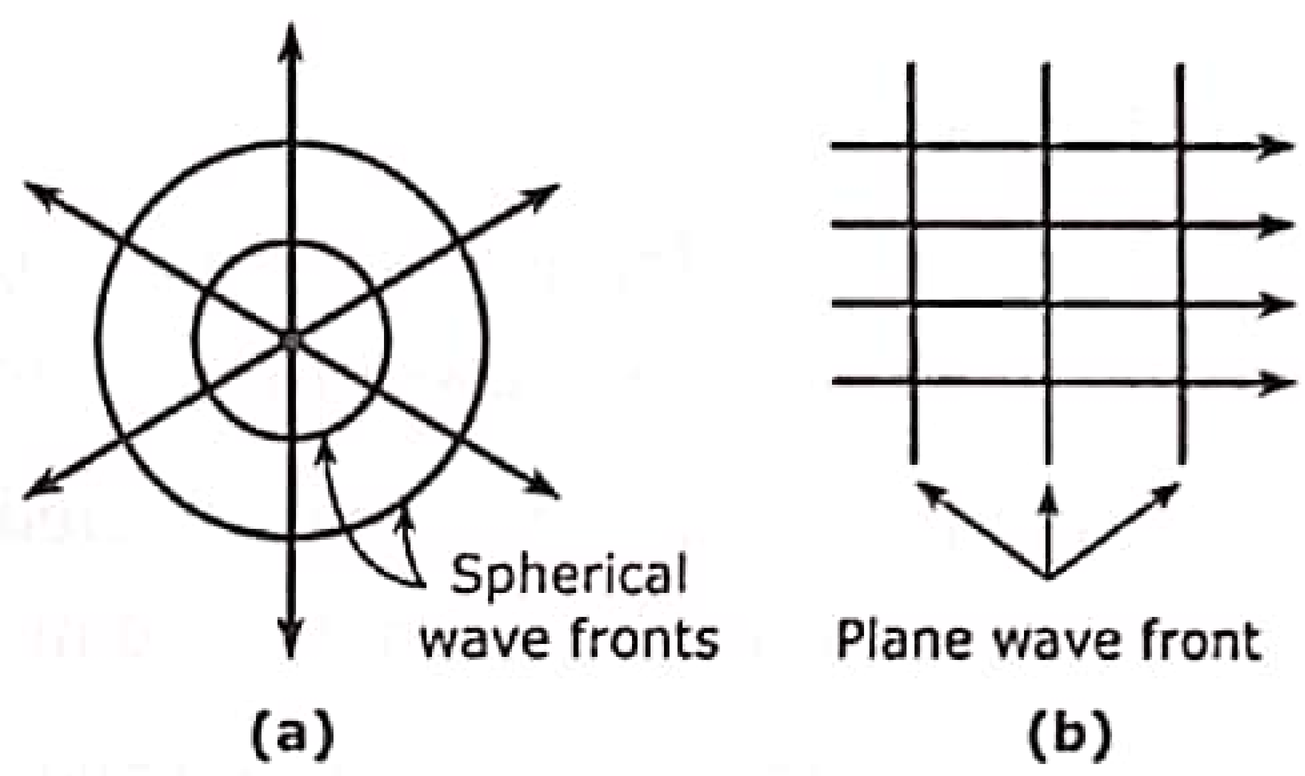

Fig. 1.1 : Wave fronts

1.2.2 : Reflection and Refraction

(A) Reflection

When a beam of light is incident on an interface separating two optical media the light is partly reflected into the first medium and partly transmitted into the second medium. This phenomenon is called reflection of light.

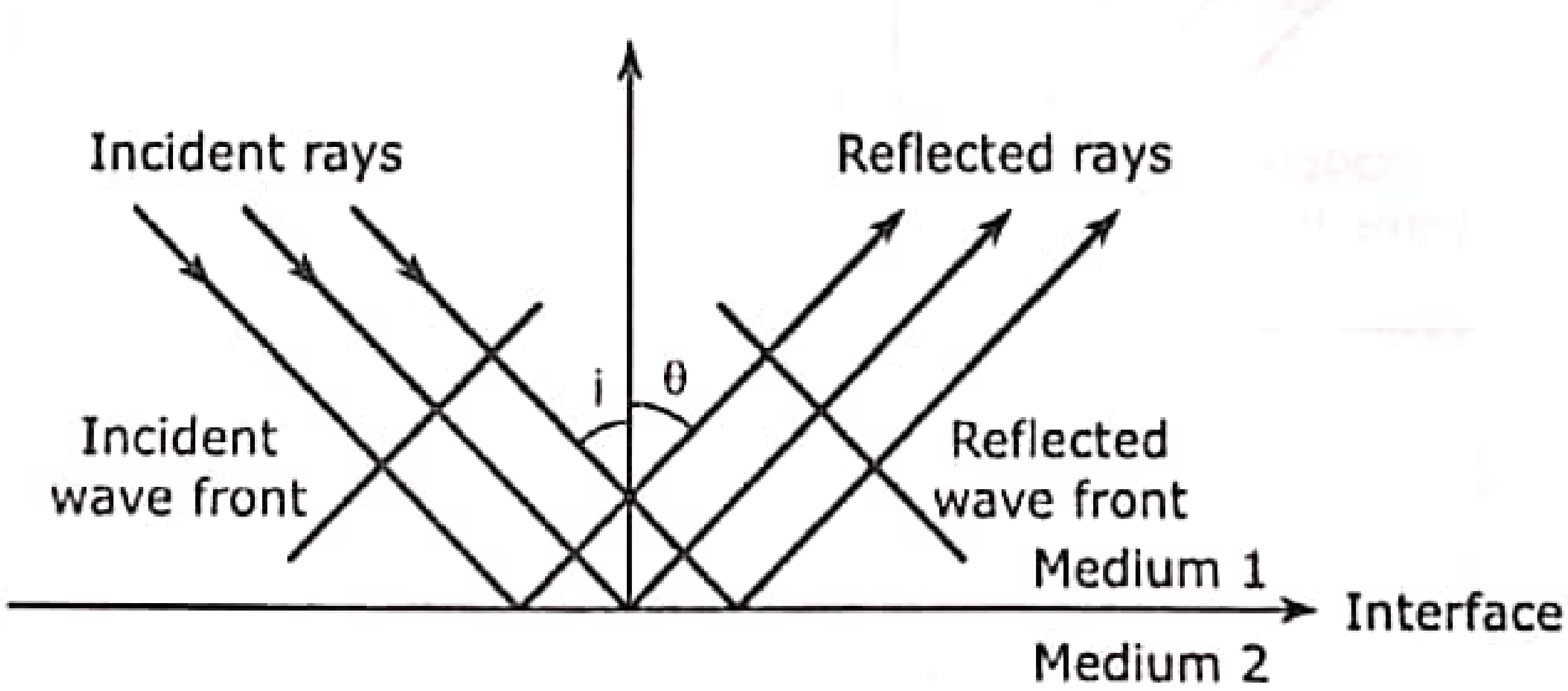

Fig. 1.2

In Fig. 1.2, it is seen that a plane wave is incident at the angle of incidence, 'i' on the interface and is reflected at the angle of reflection, 'θ'. The phenomenon follow the laws of reflection, *i.e.*,

(i) The incident beam, the reflected beam and the normal to the interface, all l the same plane called the plane of incidence.

(ii) The angle of incidence, i and the angle of reflection, θ are equal.

According to Huygen's theory a beam of light is represented by a series of w fronts which are perpendicular to the beam. Hence, the phenomenon of reflection is change of direction of a wave front at an interface between two transponent media, as in Fig. 1.2.

(B) Refraction

Refraction is defined as the bending of light wave as it passes from one transpa medium to another. The laws of refraction state that

(i) the incident ray, the refracted ray and the normal drawn perpendicular to interface between the two media, lie in the same plane.

(ii) the angle of incidence, i and the angle of refraction, r obey Snell's law, *i.e.*

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1}$$

where μ_1 and μ_2 are the refractive indices of the first and the second m respectively.

The bending of wave front during the incidence and refraction of light wave is s in Fig. 1.3.

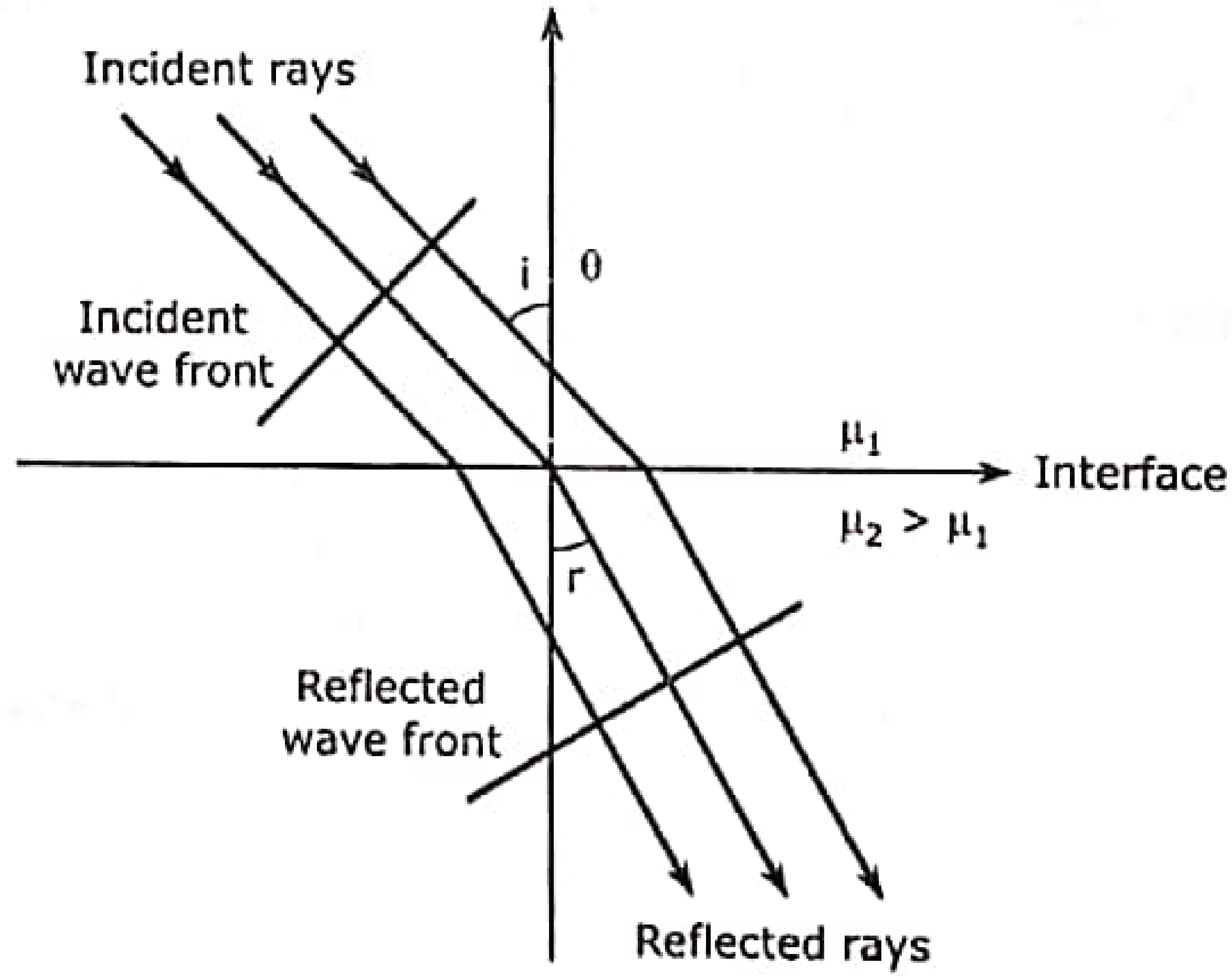

Fig. 1.3

1.2.3 : Huygen - Fresnel Principle of Diffraction

Diffraction is defined as the bending of light around an opening or an obstacle.

Hyygen-Fresnel principle states that during diffraction every point on a wavefront is a source of secondary wavelents. These wavelents spread out in the forward direction, at the same speed as the source wave. The new wavefront, thus formed, is a line tangent to all the wavelents. This is shown in Fig. 1.4.

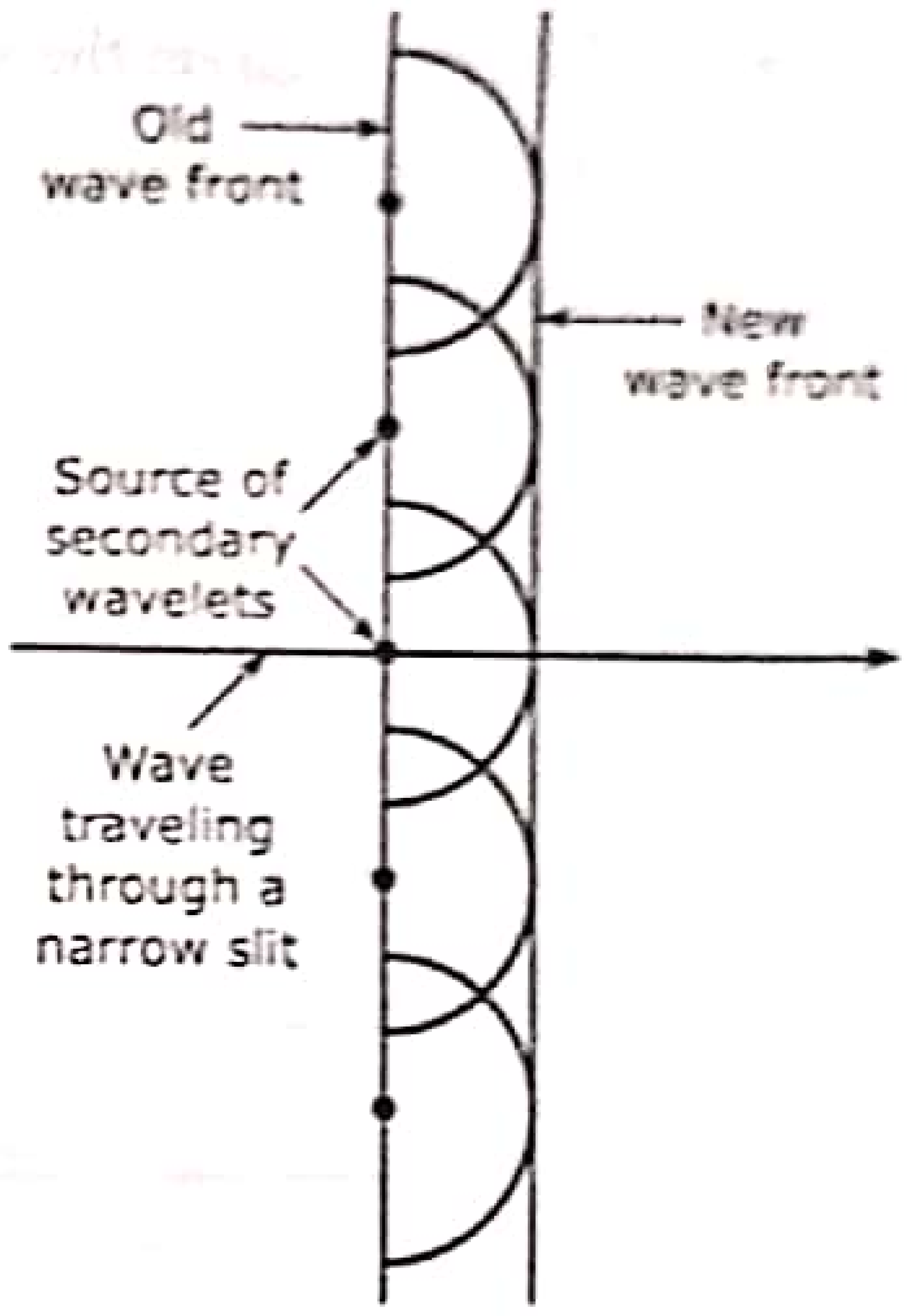

Fig. 1.4

1.2.4 : Types of Diffraction

The diffraction phenomena are divided into two categories as follows :

(i) **Fresnel diffraction** in which the source of light and the screen are, in general, at a finite distance from the obstacle as shown in Fig. 1.5 (a).

(ii) **Fraunhoffer diffraction** in which the source of light and the screen are placed at an infinite distance from the obstacle as shown in Fig. 1.5 (b).

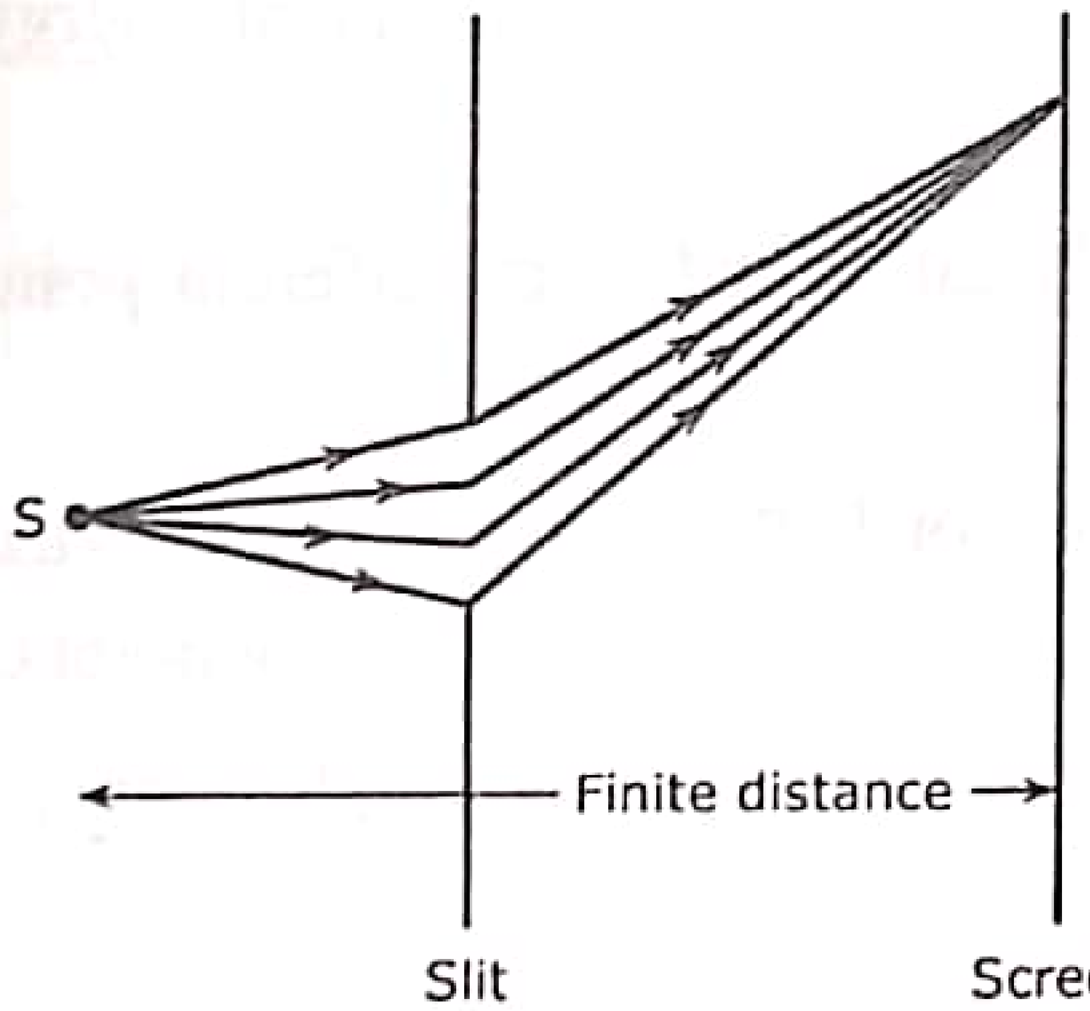

Fig. 1.5 (a)
Fresnel Diffraction

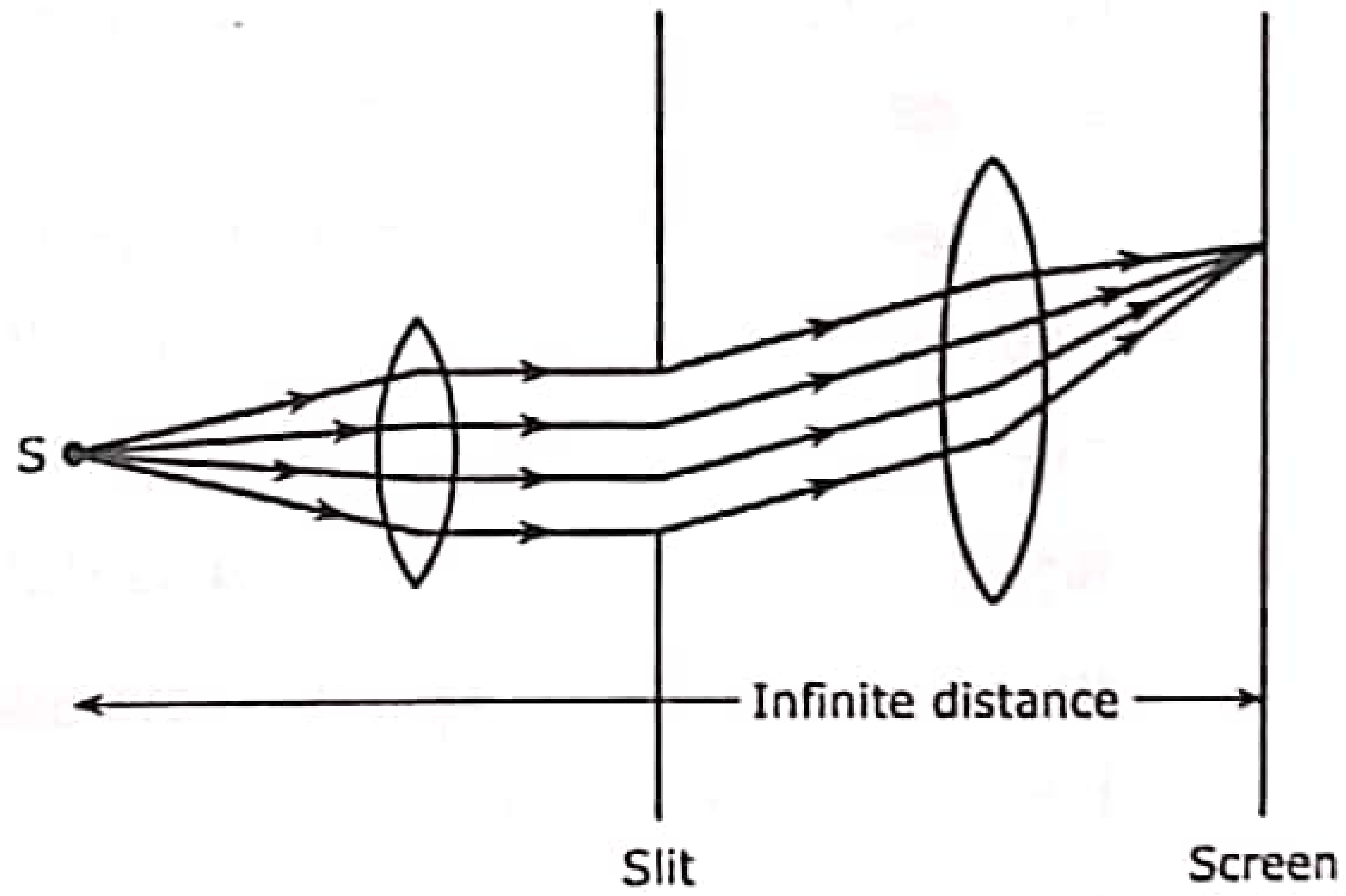

Fig. 1.5 (b)
Fraunhoffer Diffraction

1.3 Fraunhoffer Diffraction

- Consider a plane wave front (parallel beam of monochromatic rays) incident on a slit.

- Every point on the slit is a source of secondary wavelets according to Huyge principle. These wavelets interfere with the wavelets emanating from oth points, in a way, as is shown in Fig. 1.6.

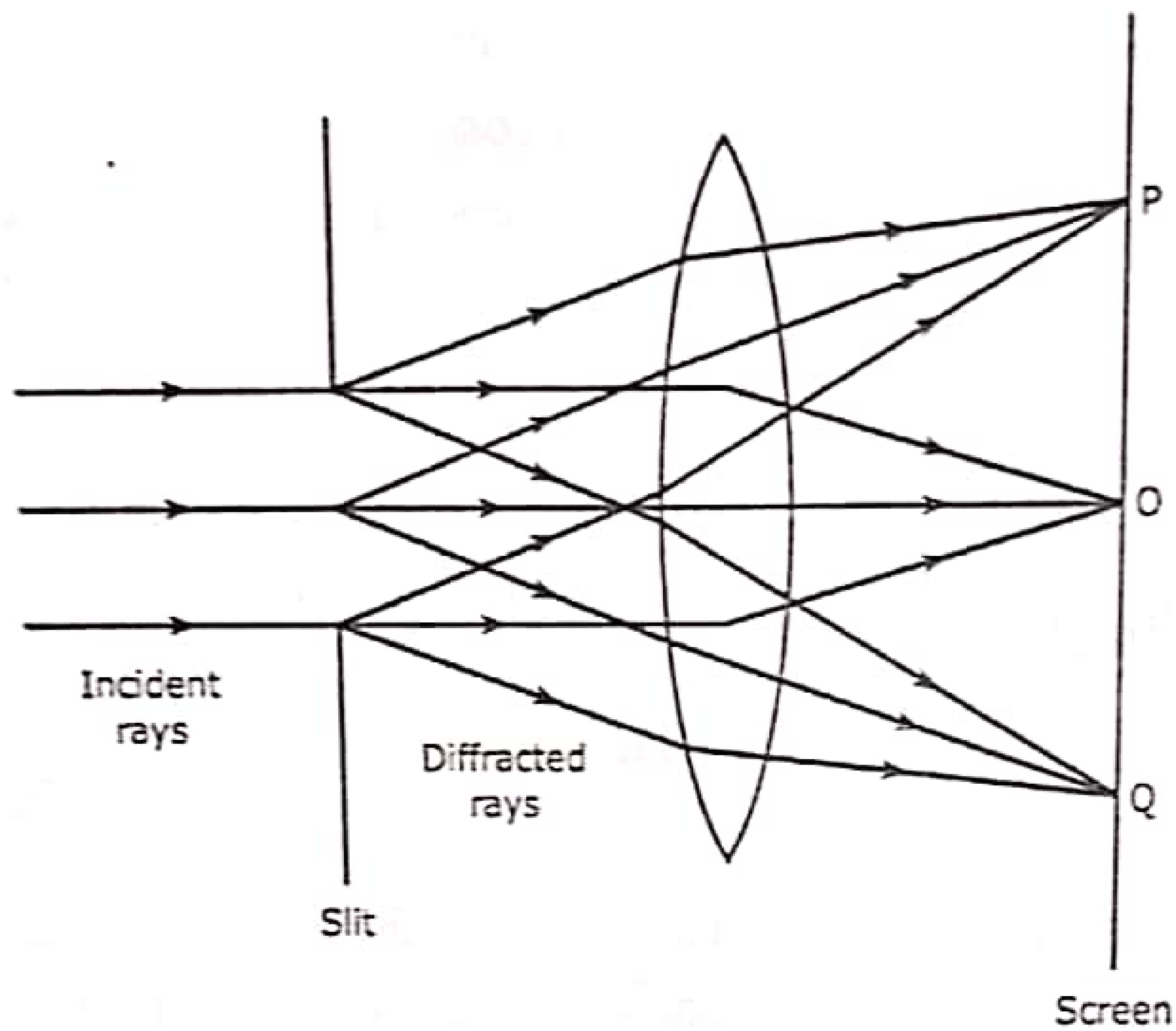

Fig. 1.6 : Fraunhoffer diffraction

- The undiffracted rays travel straight along the shortest path and interfere at th centre O of the diffraction pattern. This point receives maximum optical energ and hence it is the most intense point.
- It is seen here that a group of rays parallely diffracted from different poi sources interfere at a single point on the screen.

  The sets of parallel rays which travel identically on both sides of the undeviate rays meet at two equidistant points P and Q on both sides of the central point C
- Similarly, more sets of parallel rays interfere at various points equidistantly o both sides of point O.
- Depending on the type of interference taking place the points appear as maxima or minima. At all these points a sharp image of the slit is formed. The image has maximum brightness at the centre followed by secondary maxima with intensities gradually decreasing with distance. This intensity distribution on the screen is known as the Fraunhoffer diffraction pattern which is actually the interference pattern of the diffracted waves.

1.3.1 : Fraunhoffer Diffraction at a Single Slit

- Consider a plane wave incident on a slit of width b.
- The slit is assumed to consist of n number of point sources of secondary wavelets. Let the point sources A_1, A_2, ..., A_n be separated by a constant distance, Δ.
- To study the diffraction pattern produced on the screen (on the focal plane of the lens) calculate the intensity at any arbitrary point P.

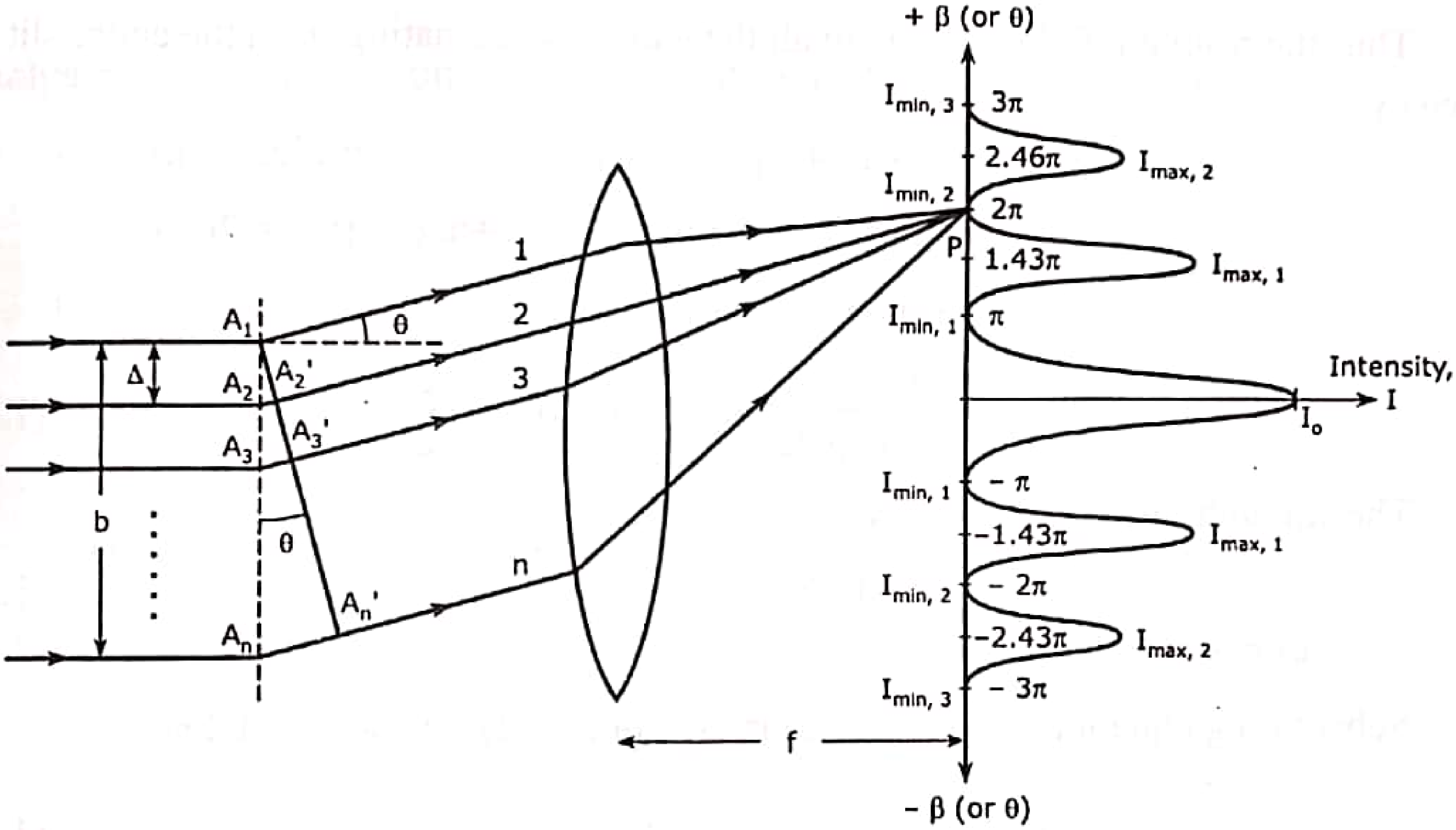

Fig. 1.7 : Fraunhoffer's Single slit diffraction

- Consider a set of parallel rays that interfere at point P and produces an intensity.
- For an incident plane wave the secondary waves emanating from points A_1, A_2, ..., A_n are in phase. By dropping a perpendicular from A_1 to different wavelets the path differences $A_2 A_2'$, $A_3 A_3'$, ..., $A_n A_n'$ can be calculated.

If θ is the angle of diffraction, the path difference between ray 1 and ray 2 is

$$A_2 A_2' = \Delta \sin \theta$$

It is known that a path difference of $\lambda / 2$ produces a phase difference of π. Hence, the path difference of $A_2 A_2$ introduces a phase difference given by

$$\phi = \frac{2\pi}{\lambda} \Delta \sin \theta \qquad \text{.............. (1.1)}$$

Thus if the wavelet which reaches point P after being emanated form point ... written as

$$e_1 = a \cos \omega t$$

the wavelet emanating from A_2 with a phase difference of ϕ is written as

$$e_2 = a \cos (\omega t + \phi)$$

and that of the wavelet from A_3 is

$$e_3 = a \cos (\omega t + 2\phi)$$

Thus the resultant field at P due to all the wavelets emanating from the entire ... given by

$$E = e_1 + e_2 + e_3 + ... + e_n$$

$$= a \cos \omega t + a \cos (\omega t + \phi) + a \cos (\omega t + 2\phi) + ...$$

$$+ a \cos [\omega t + (n-1)\phi]$$

$$= a \frac{\sin (n\phi/2)}{\sin (\phi/2)} \cos \left[\omega t + (n-1)\frac{\phi}{2} \right] \quad \text{............... (}$$

The slit width can be written as

$$b = (n-1)\Delta \simeq n\Delta \quad \text{............... (}$$

as n is very large.

Substituting equation (1.1) and (1.3) in equation (1.2) it is obtained that

$$E = a \frac{\sin \beta}{\sin (\beta/n)} \cos (\omega t + \beta) \quad \text{............... (}$$

where $$\beta = \frac{\pi b \sin \theta}{\lambda} \quad \text{............... (}$$

As n is very large, $n \to \infty$, $\frac{\beta}{n} \to 0$ and

$$\lim_{(\beta/n) \to 0} \sin \left(\frac{\beta}{n} \right) = \frac{\beta}{n} \quad \text{............... (}$$

Hence, equation (1.5) becomes

$$\boxed{E = A \frac{\sin \beta}{\beta} \cos (\omega t + \beta)} \quad \text{............... (1}$$

where A = na. This is the resultant field at the point P with an amplitude,

$$E_A = A\frac{\sin\beta}{\beta} \quad \text{.............. (1.8)}$$

Hence, the intensity at P is given by

$$I = |E_A|^2 = A^2\frac{\sin^2\beta}{\beta^2} \quad \text{.............. (1.9)}$$

which is written as

$$\boxed{I = I_o\frac{\sin^2\beta}{\beta^2}} \quad \text{.............. (1.10)}$$

Analysis of the Diffraction Pattern : Positions of Maxima and Minima

(1) Central maximum

This is produced by the undiffracted rays for which $\theta = 0$.

For $\theta = 0$, equation (1.5) becomes

$$\beta = \frac{\pi b \sin\theta}{\lambda} = 0$$

Since, $\lim\limits_{\beta\to 0}\frac{\sin\beta}{\beta} = 1$

the intensity at the central point of the diffraction pattern is

$$I_{central} = I_o \lim_{\beta\to 0}\frac{\sin^2\beta}{\beta^2} = I_o = I_{max} \quad \text{.............. (1.11)}$$

(2) Secondary Maxima

As the secondary maxima are identically spaced on both sides of the central maxima

$$\frac{dI}{d\theta} = 0$$

Referring to equation (1.10) this can be written as

$$\frac{dI}{d\beta} = 0 \quad \text{.............. (1.12)}$$

Using equation (1.10), here it is found that

$$\tan\beta = \beta \quad \text{.............. (1.13)}$$

which is true for $\beta = 0, \pm 1.43\pi, \pm 2.46\pi$, ..., so on.

Here, $\beta = 0$ corresponds to the central maximum, as already shown in equation (1.11). Hence, secondary maxima are produced at angles $\beta = \pm 1.43\pi, \pm 2.46\pi$, ...

(3) Secondary Minima

The central and secondary maxima are separated by the secondary minima. At minima the intensity is zero. Hence,

$$I = I_0 \frac{\sin^2 \beta}{\beta^2} = 0$$

i.e., $\sin \beta = 0$

This is true for $\beta = m\pi$ with $m = 0, \pm 1, \pm 2, ...$ .... (1.

Here $m = 0$, *i.e.*, $\beta = 0$ corresponds to the central maximum. Hence, the second maxima occur at the positions for which $\beta = \pm \pi, \pm 2\pi, \pm 3\pi$, ... and so on. The positi of all the maxima and minima are shown in Fig. 1.7.

1.3.2 : Fraunhoffer Diffraction at N slits

Consider the diffraction of a plane wave by a system of N slits, each of width b. T distance between any two consecutive slits is assumed to be d. Each slit is assumed consist of n equally spaced point sources with spacing Δ. It is already found that the fi at P due to the first slit is

$$E_1 = A \frac{\sin\beta}{\beta} \cos(\omega t + \beta)$$ .... (1.

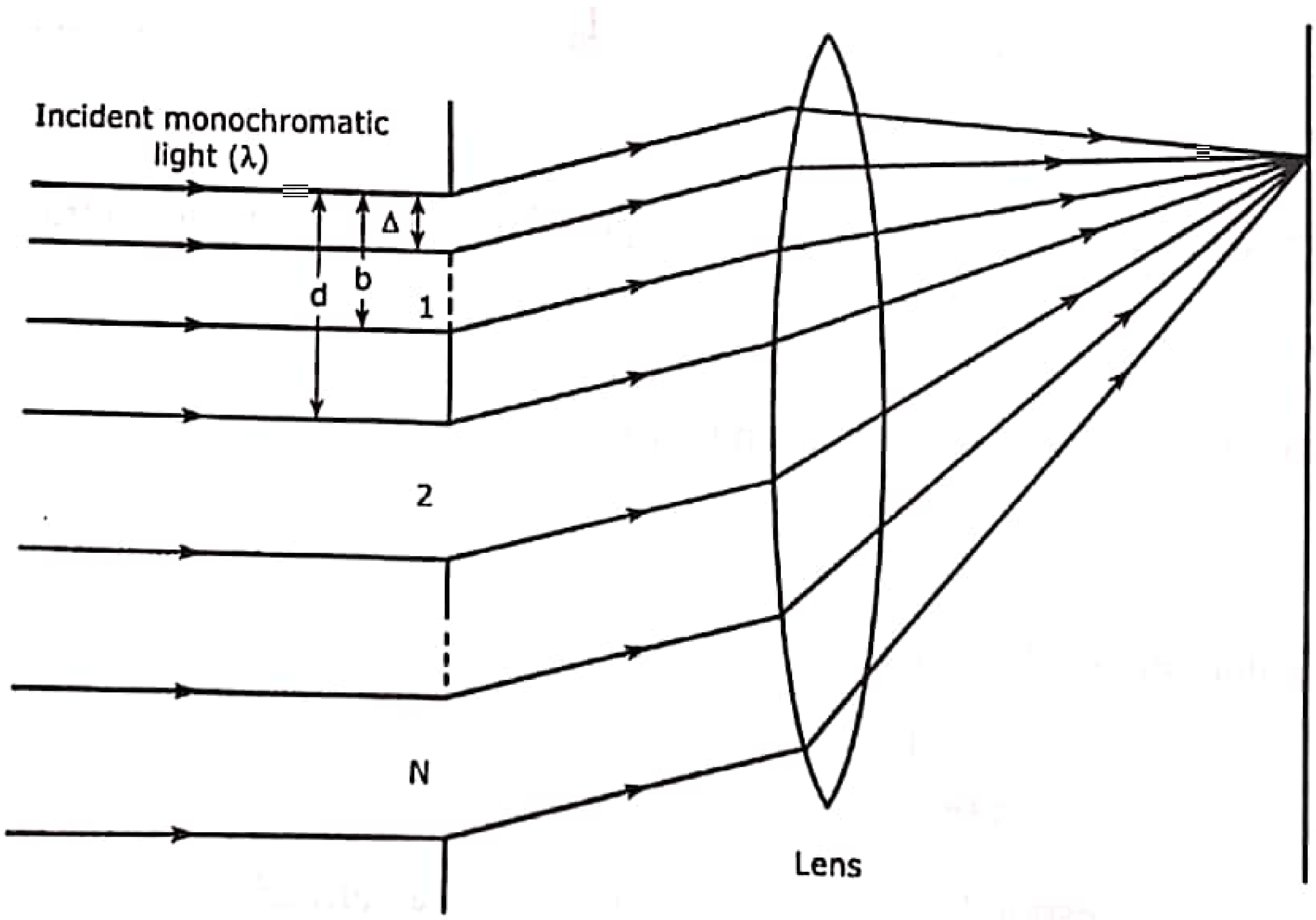

Fig. 1.8 : Fraunhoffer Diffraction at N slits

The resultant field at P due to the wavelets emanating from the system of N slits will essentially be the sum of N fields as

$$E = E_1 + E_2 + ... + E_n$$

$$= A\frac{\sin\beta}{\beta}\cos(\omega t+\beta) + A\frac{\sin\beta}{\beta}\cos(\omega t+\beta+\psi) + ...$$

$$..... + A\frac{\sin\beta}{\beta}\cos[\,\omega t+\beta+(N-1)\,\psi\,]$$

$$E = A\frac{\sin\beta}{\beta}\cdot\frac{\sin N\gamma}{\sin\gamma}\cos[\,\omega t+\beta+(N-1)\gamma\,] \qquad .... (1.16)$$

The corresponding intensity distribution is

$$I = I_o\frac{\sin^2\beta}{\beta^2}\cdot\frac{\sin N\gamma}{\sin\gamma} \qquad .... (1.17)$$

where $I_o\frac{\sin^2\beta}{\beta^2}$ represents the intensity distribution produced by a single slit.

Positions of Maxima and minima

(1) Principal maxima

For N very large, a mathematical result is

$$\lim_{\gamma\to m\pi}\frac{\sin N\gamma}{\sin\gamma} = \pm N \qquad .... (1.18)$$

Thus, the intensity for the principal maxima is found from equation (1.17) as

$$I = N^2 I_o\frac{\sin^2\beta}{\beta^2}$$

Hence, the condition for principal maxima is

$$\gamma = m\pi, \qquad m = 0, 1, 2, ...$$

i.e., $$d\sin\theta = m\lambda \qquad .... (1.19)$$

Central principal maxima occurs for m = 0 and $\theta = 0$ *i.e.*, $\rho = 0$ and $\beta = 0$.

Here $\lim_{\beta\to 0}\frac{\sin\beta}{\beta} = 1$ which gives

$$\boxed{I = N^2 I_o = I_{max}}$$ .... (1

the maximum intensity observed at the central principal maximum.

The other principal maxima, which are equispaced on both sides of the cen principal maxima are observed for m = 1, 2, 3, ...

From equation (1.5) and (1.20), it can be written as

$$\beta = \frac{\pi b \sin\theta}{\lambda} = \frac{\pi b}{\lambda}\left(\frac{m\lambda}{d}\right) = \left(\frac{b}{d}\right) m\pi$$ .... (1

the values of β for which the principal maxima are observed.

From equation (1.101), it is found that

$$m \leq \frac{d}{\lambda}, \quad \text{since } \sin\theta \leq 1$$

Hence, **there will be a finite number of principal maxima.**

(2) Secondary Minima

From equation (1.17) it is seen that for zero intensity

$$\sin N\gamma = 0 \quad \text{for } N\gamma = p\pi \quad \text{for } \pi = 1, 2, 3, ...$$

and $$\sin\beta = 0 \quad \text{for } \beta = n\pi \quad \text{for } n = 1, 2, 3 .....$$

Here $N\gamma = 0$ and $\beta = 0$ both results into $\theta = 0$ which corresponds to the cen principal maximum.

Hence, the condition for secondary minima is

$$d \sin\theta = \frac{p}{N}\lambda, \quad p = 1, 2, 3, ...$$ .... (1.

and $$b \sin\theta = n\lambda, \quad n = 1, 2, 3, ...$$ .... (1.

Expanding equation (1.22), it is found that

$$d \sin\theta = \frac{\lambda}{N}, \frac{2\lambda}{N}, ..., \frac{(N-1)\lambda}{N}, \lambda.$$ .... (1.

in which $d \sin\theta = \lambda$ corresponds to the N^{th} principal maximum whereas d sin θ = corresponds to the central principal maximum Hence, from equation (1.90), it is obvi that there are (N – 1) secondary minima in between any two consecutive principal maxi

In a practical set up the diffraction pattern consists of the maxima for which condition is given by equation (1.19) as

$$(a + b) \sin\theta = m\lambda$$ .... (1.

known as the ***diffraction formula.***

1.4 Diffraction Grating and its Characteristics

- A diffraction grating is a very important application of diffraction. This is a series of a very large number of extremely narrow parallel slits separated by opaque spaces. There are two types of grating :
  (i) Transmission grating through which light is transmitted, and
  (ii) Reflection grating from which light is reflected.
- Transmission gratings are made by drawing very fine equispaced rulings on the surface of a very good quality glass plate using a diamond tip. The rulings act as the opaque spaces and the gaps of equal width between the rulings act as a system of slits.

1.4.1 : Different Characteristics of a Plane Transmission Grating

(A) Grating element : (a + b)

- If the width of each ruling is a and the width of each slit is b, the length (a + b) is called the *grating element*.
- As there are very large number of lines, say N lines, there are (N – 1) slits present on any grating.

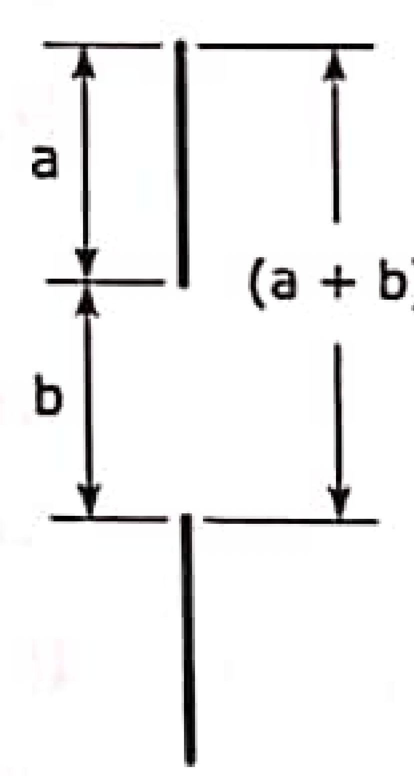

Fig. 1.9

Number of lines, N ≃ number of slits (N – 1).

$$\text{The number of lines/cm} = \frac{1}{a+b}$$

and
$$\text{The number of lines /inch} = \frac{2.54}{a+b}$$

since 1 inch = 2.54 cm.

The dimension of the grating element (a + b) is of the order of the wave length of visible light.

(B) Missing Orders : Absent Spectra

Sometimes a few maxima disappear from the diffraction pattern. This can be explained as follows :

From equation (1.103) and (1.105), it is found that for a grating of opaque space width a, slit width b and grating element d = a + b,

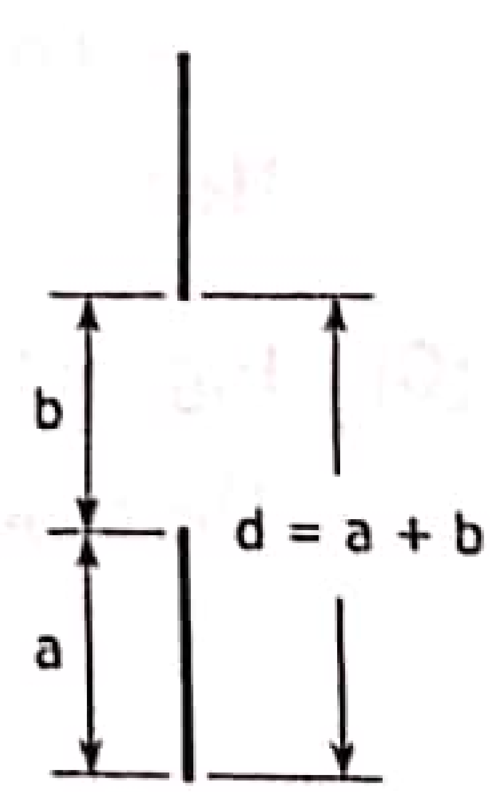

Fig. 1.10

Condition for maxima : $(a + b) \sin\theta = m\lambda$ .... (1.26-a)

Condition for minima : $b \sin\theta = n\lambda$ .... (1.26-b)

When some of the maxima satisfy the condition for minima absent spe appears.

Lets assume that the absent maxima satisfy equation (1.26-a). Hence, m is the o of the absent maxima and n (= 1, 2, 3, ...) is the order of the regular minima. The unkn orders of the absent maxima are found by dividing equation (1.106-a) by equation (1.2 as

$$\frac{a+b}{b} = \frac{m}{n}$$

or

$$\boxed{m = n\left(\frac{a+b}{b}\right)} \quad \text{.............. (1}$$

This can be explained as follows :

(i) If the slit width is same as the width of the ruling, a = b and equation (1 becomes

$$m = 2n$$

The regular orders of minima are n = 1, 2, 3, ...

Hence, the maxima of orders

$$m = 2, 4, 6, ...$$

will be missing from the spectrum. This means that all the even order ma remain absent in the grating spectrum.

(ii) If the width of the ruling is double the slit width a = 2b and equation (1 becomes

$$m = 3n$$

Substituting the regular orders of minima, n = 1, 2, 3, ... here it is found

$$m = 3, 6, 9, ...$$

i.e., the 3^{rd}, 6^{th}, 9^{th}, ... order maxima will remain absent in the grating spec

Hence, it is found that the ***absent spectrum depends on the dimensions of the gra***

(C) Highest Possible Orders of Maxima in a Grating Spectrum : m_{max}

The condition for maxima is given by

$$(a+b)\sin\theta = m\lambda$$

For a given monochromatic light of wavelength λ incident on a given plane transmission grating of grating element (a + b),

$$(a+b)\sin\theta_{max} = m_{max}\lambda$$

$$\boxed{m_{max} = \frac{a+b}{\lambda}} \qquad \text{.............. (1.28)}$$

as $\sin\theta_{max} = 1$

This can be explained with a few examples, as follows :

(i) If the grating element is less than the incident wavelength *i.e.*, **$(a + b) < \lambda$**,

$$m_{max} < 1$$

This means that only the central maximum is visible.

(ii) If the grating element is less than double the incident wavelength *i.e.*, **$(a + b) < 2\lambda$**,

$$m_{max} < 2$$

This means that the central maximum and the two first order maxima are visible.

It is known that the grating element is given by

$$a+b = \frac{1}{\text{No. of lines/cm}}$$

Hence,

$$m_{max} = \frac{1}{\lambda \times \text{No. of lines/cm}} \qquad \text{.............. (1.29)}$$

Therefore, the larger the number of rulings on the grating surface, the smaller is the number of visible orders in the grating spectrum

1.5 Resolving Power of a Diffraction Grating

When two objects are placed very close to each other or kept at a large distance human eye cannot distinguish them. Telescopes or microscopes are required to be used to view them. These optical instruments receive the rays diffracted by these objects and produce the corresponding diffraction patterns consisting of various maxima and minima. The objects can be viewed distinctively if their diffraction patterns are distinguishable. The ability of an optical instrument to produce just separate diffraction patterns of two close objects is called its ***resolving power***.

1.5.1 : Rayleigh's Criteria of Resolution

Two close spaced point sources of light are said to be just resolved by an o instrument only if the first principal maximum of the diffraction pattern due to one s coincides with the first minimum of the diffraction pattern due to the other source., a in Fig. 1.11.

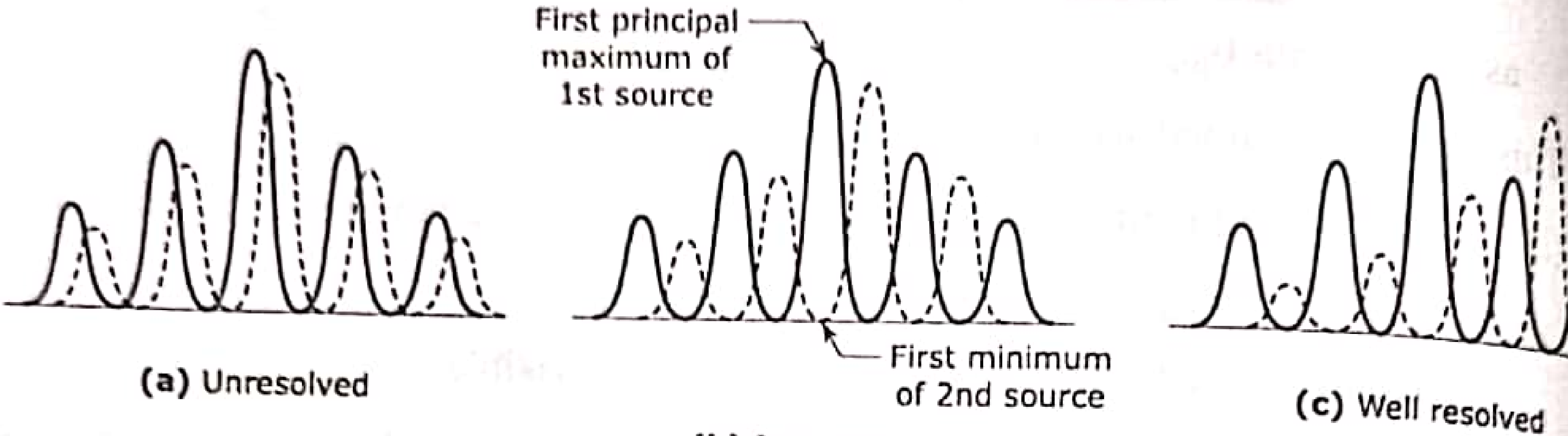

Fig. 1.11 : Rayleigh's Criteria

1.5.2 : Resolving Power of a Grating

To determine the resolving power of a grating consider a beam of light conta two wavelengths λ and $\lambda + d\lambda$ very close to each other, incident on the grating surfa grating element (a + b).

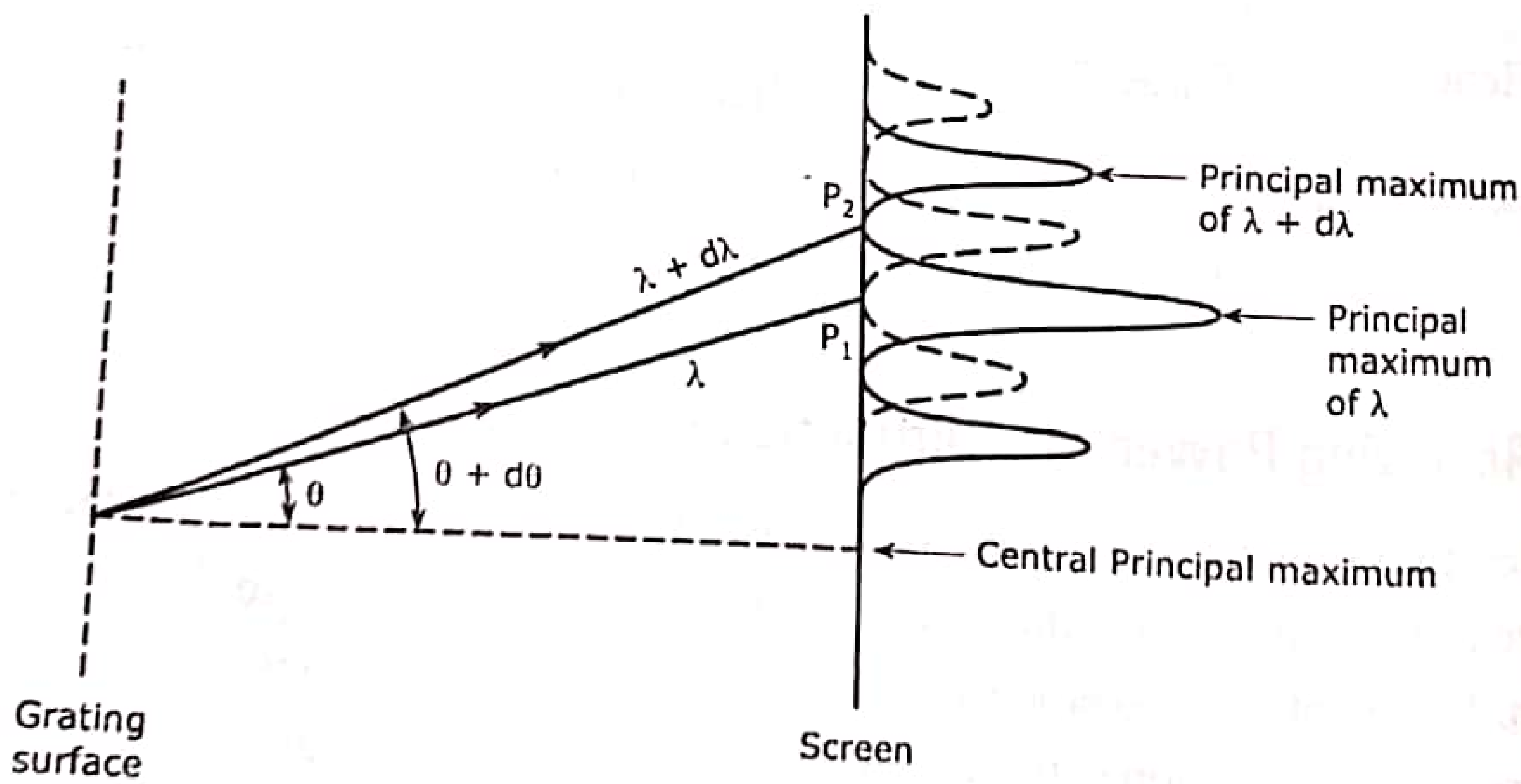

Fig. 1.12 : Resolving power of a grating

As seen in Fig. 1.12, the wavelength λ after being diffracted through an ang strikes the screen at point P_1 where the first principal maximum of its spectrum app

The wavelength $\lambda + d\lambda$ is diffracted through an angle $\theta + d\theta$ and strikes the screen at point P_2 where its first principal maxima is produced (shown with dotted curve).

According to Rayleigh's criteria if the two waves of wavelengths λ and $\lambda + d\lambda$ are to be resolved by the grating, at point P_2, the first minimum due to λ should coincide with first principal maximum of $\lambda + d\lambda$.

The conditions for the m^{th} order maxima for the waves of wavelengths λ and $\lambda + d\lambda$ which are diffracted through θ and $\theta + d\theta$ respectively are given by,

$$(a + b) \sin \theta = m \lambda \quad \text{and} \qquad \text{.............. (1.30)}$$

$$(a + b) \sin (\theta + d\theta) = m (\lambda + d\lambda) \qquad \text{.............. (1.31)}$$

The distance between the central maxima due to wavelengths λ and $\lambda + d\lambda$ is given by

$$P_1 P_2 = (a + b) \sin (\theta + d\theta) - (a + b) \sin \theta$$

$$= m (\lambda + d\lambda) - m \lambda$$

$$P_1 P_2 = m\, d\lambda \qquad \text{.............. (1.32)}$$

On the other hand, P_1P_2 is the distance between the central maximum and the first minimum in the spectrum of the wave of wavelength λ.

From equation (1.22) the condition for minima is given by

$$(a + b) \sin \theta = \frac{p}{N} \lambda, \qquad p = 1, 2, 3, \ldots.$$

which when expanded becomes

$$(a + b) \sin \theta = \frac{\lambda}{N}, \frac{2\lambda}{N}, \ldots. \qquad \text{.............. (1.33)}$$

where $(a + b) \sin \theta = 0$ .... (1.34)

corresponds to the central maximum.

From equations (1.33) and (1.34), it is understood that from the central maximum, the first minimum occurs at a distance λ / N, the second minimum occurs at distance $2\lambda / N$, and so on.

Here P_1P_2 is the distance between the central maximum and the first minimum in the spectrum of λ.

$$P_1 P_2 = m\, d\lambda = \frac{\lambda}{N} \qquad \text{.............. (1.35)}$$

The resoling power of a grating.

$$\text{R.P.} = \frac{\lambda}{d\lambda} = mN \qquad \text{.........(}$$

is defined as the ratio of the wavelength of any spectral line (say, λ of wavelength) difference in the wavelength (dλ) between the former and the neighboring line (i.e. wavelength λ + dλ) such that the two lines appear to be just resolved.

Here, N is the minimum number of lines required on the grating surface to resolve the wavelengths λ and λ + dλ.

From equation (1.36) it is obvious that **by increasing the number of rulings on grating surface the resolving power can be improved.**

1.5.3 : Polychromatic Grating Spectra

- Consider a plane transmission grating illuminated by a beam of white light conta seven wavelengths ranging from violet to red.
- Every wavelength will exhibit a separate diffraction spectrum containing the ce principal maximum and higher order principal maxima with gradually decre intensities on its both sides.
- As the dispersive power of a grating is given by $\frac{d\theta}{d\lambda}$, the larger the wavelength more is the dispersion.

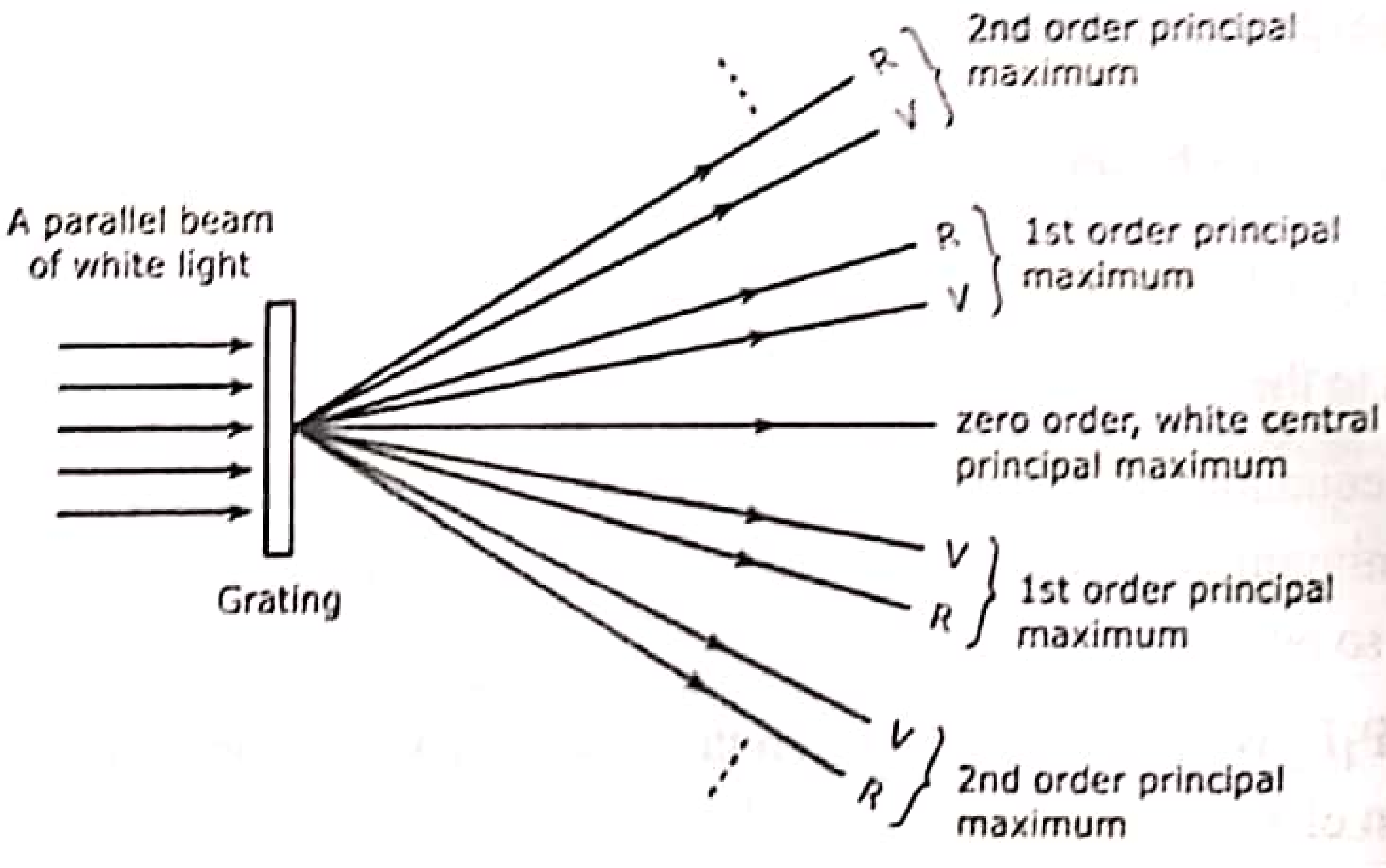

Fig. 1.13 : Polychromatic Grating Spectra

- Hence, every principal maxima will appear to split into seven different colo according to their λ values as shown in Fig. 1.13.

- The individual grating spectrum of each colour consists of a central principal maximum. Hence, at the central principal maxima all the colours overlap producing a white spot.
- In the diffraction spectrum a central white spot is found with coloured bands (ranging from violet to red outwards) equally spaced on its both sides with gradually diminishing intensities.

1.6 Application of Diffraction Grating : Determination of Unknown Wavelength of Light by Diffraction Grating

Diffraction grating has wide use in determination the wavelength of light by using a spectrometer in Laboratory.

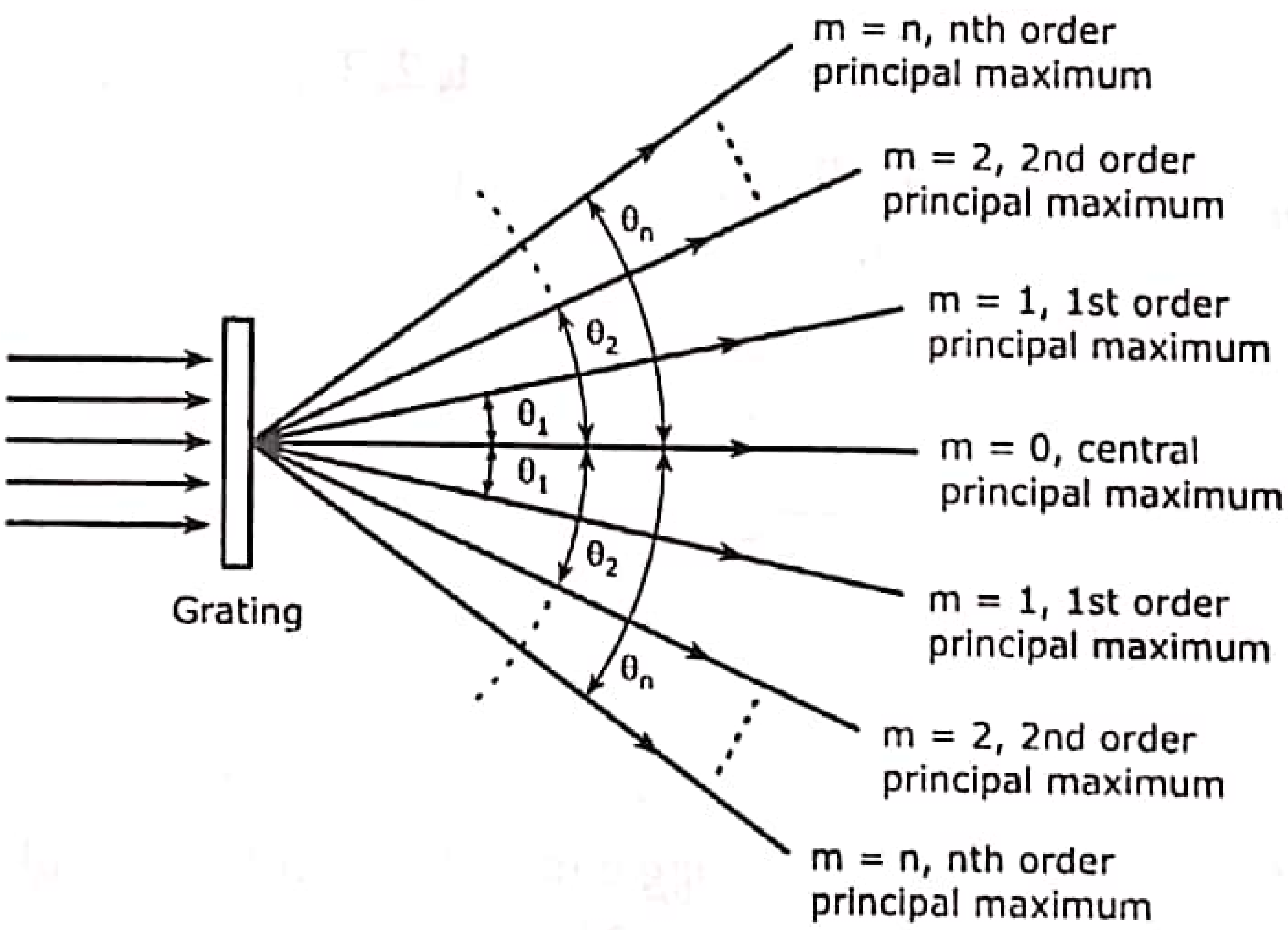

Fig. 1.14 : Polychromatic grating spectra

As seen in Fig. 1.14 a parallel beam of monochromatic light of wavelength is made o fall on the grating surface of grating element (a + b). The diffraction pattern is formed on the screen, consisting of the central principal maximum and higher order principal naxima with gradually decreasing intensity on its both sides.

The telescope of the spectrometer is adjusted to view the central principal maximum m = 0). Then the telescope is rotated through an angle $2\theta_1$ to view the two first order principal maxima on either side of the central spot. For first order diffraction maxima, it is vritten as

$$(a + b) \sin \theta_1 = \lambda$$

Knowing the grating element (a + b), λ can be determined.

The same method can be used for any order of diffraction.

1.7 Solved Problems

Problem 1

A parallel beam of monochromatic light is incident on a plane transmission gra having 3000 lines/cm. A third order diffraction is observed at 30°. Calculate the wavele of the line. **(M.U. May 2007; Dec. 2006)**

Solution :

Data : $a + b = \frac{1}{3000}$ cm, $n = 3$, $\theta = 30°$.

Formula : $(a + b) \sin\theta = n\lambda, \quad n = 1, 2, 3, ...$

Calculation :

$$\lambda = \frac{a+b}{n} \sin\theta$$

$$= \frac{1}{3000 \times 3} \times \sin 30°$$

$$= \frac{1}{9000 \times 2} = 5.555 \times 10^{-5} \text{ cm}$$

Result : Wavelength = 5555 A°.

Problem 2

In a plane transmission grating, the angle of diffraction for second order princ maximum for the wavelength 5×10^{-5} cm is 35°. Calculate the number of lines / cm the grating surface. **(M.U. May 2013, 17)**

Solution :

Data : $\theta = 35°$, $\lambda = 5 \times 10^{-5}$ cm, $n = 2$.

Formula : $(a + b) \sin\theta = n\lambda, \quad n = 1, 2, 3, ...$

$$a + b = \frac{1}{\text{No. of lines / cm}}$$

Calculation :

$$(a + b) = \frac{n\lambda}{\sin\theta}$$

$$\therefore \quad (a+b) = \frac{2\times5\times10^{-5}}{\sin 35^\circ} = 17.43\times10^{-5}\text{ cm}$$

$$\text{No. of lines/cm} = \frac{1}{17.43\times10^{-5}} = 5737$$

Result : No. of lines/cm = 5737.

Problem 3

Monochromatic light of wavelength 6560 A° falls normally on a grating 2 cm wide. The first order spectrum is produced at an angle of 18°14' from the normal. Calculate the total numbers of lines on the grating.

(M.U. May 2013, 18) (5 m)

Solution :

Data : $\lambda = 6560\text{ A}^\circ = 6560\times10^{-8}\text{ cm},$

$\theta = 18^\circ\ 14' = 18.84^\circ,\ \text{width} = 2\text{ cm},\ n = 1.$

Formula : $(a+b)\sin\theta = n\lambda, \qquad n = 1, 2, 3, ...$

$$a+b = \frac{1}{\text{No. of lines / cm}}$$

Calculation : $$(a+b) = \frac{n\lambda}{\sin\theta} = \frac{6560\times10^{-8}}{\sin 18.84^\circ}$$

$$= 2.03\times10^{-4}\text{ cm.}$$

$$\text{No. of lines/cm} = \frac{1}{a+b} = \frac{1}{2.03\times10^{-4}} = 4926$$

$$\text{Total No. of lines} = 4926\times2 = 9852$$

Result : Total No. of lines on the grating surface = 9852.

Problem 4

In plane transmission grating the angle of diffraction for the second order principal maxima for the wavelength 5×10^{-5} cm is 35°. Calculate the number of lines/cm on the diffraction grating.

(M.U. Nov..2016) (5 m)

Solution :

Data : $\lambda = 5\times10^{-5}\text{ cm},\ \theta = 35^\circ,\ n = 2.$

Formula : $(a+b)\sin\theta = n\lambda, \qquad n = 1, 2, 3, ...$

$$\frac{1}{a+b} = \text{Number of lines/cm}$$

Calculation : $$a+b = \frac{n\lambda}{\sin\theta} = \frac{2\times5\times10^{-5}}{\sin 35°}$$

$$= 1.74 \times 10^{-4}$$

$$\text{Number of lines/cm} = \frac{1}{a+b} = \frac{1}{1.74\times10^{-4}} = 5735$$

Result : Number of lines/cm = 5735

Problem 5

In a phase transmission grating the angle of diffraction for the first order prir maximum is 20° for a wavelength of 6500 A°. Calculate the number of lines in one the grating surface.

(M.U. Dec. 2019)

Solution :

Data : $n = 1,\ \theta = 20°,\ \lambda = 6500\ A° = 6.5 \times 10^{-5}$ cm.

Formula : $(a+b)\sin\theta = n\lambda, \quad n = 1, 2, 3, ...$

$$a+b = \frac{1}{\text{No. of lines / cm}}$$

Calculation : $$(a+b) = \frac{h\lambda}{\sin\theta} = \frac{1\times6.5\times10^{-5}}{\sin 20°}$$

$$= 19.0047 \times 10^{-5}$$

$$\text{No. of lines / cm} = \frac{1}{a+b} = \frac{1}{19.0047\times10^{-5}} = 5261$$

Result : Number of lines in one cm = 5261.

Problem 6

A monochromatic light of wavelength 5×10^{-5} cm falls normally on a grating, wide. The first order maxima is produced at 18° from the normal. What are the total nu of lines on the grating?

(M.U. Dec. 2019)

Solution :

Data : $n = 1,\ \theta = 18°,\ \lambda = 5 \times 10^{-5}$ cm, width = 2 cm.

Formula : $(a+b)\sin\theta = n\lambda, \quad n = 1, 2, 3, ...$

$$a + b = \frac{1}{\text{No. of lines / cm}}$$

Calculation : $$(a + b) = \frac{n\lambda}{\sin\theta} = \frac{1 \times 5 \times 10^{-5}}{\sin 18^\circ}$$

$$= 1.618 \times 10^{-4} \text{ cm}$$

$$\text{No. of lines / cm} = \frac{1}{a+b} = \frac{1}{1.618 \times 10^{-4}} = 6180$$

Total no. of lines on the grating surface

$$= 6180 \times 2 = 12360$$

Result : Number of lines in one cm = 12360.

Problem 7

Calculate the highest order spectrum that can be obtained by a monochromatic light of wavelength 6000 A° by a grating with 6000 lines/cm.

Solution :

Data : $\lambda = 6000 \text{ A}^\circ = 6000 \times 10^{-8}$ cm,

No. of lines/cm = 6000.

Formula : $(a + b) \sin\theta = n\lambda, \quad n = 1, 2, 3, ...$

$$a + b = \frac{1}{\text{No. of lines / cm}}$$

Calculation : For given (a + b) and λ

$$\sin\theta \propto n$$

$$(a + b) \sin\theta_{max} = n_{max}\lambda$$

$$\sin\theta_{max} = 1$$

$$(a + b) = n_{max}\lambda$$

$$n_{max} = \frac{a+b}{\lambda} = \frac{1}{6000 \times 6000 \times 10^{-8}}$$

$$\therefore \quad n_{max} = 2.7$$

As $n_{max} = 2.7 < 3$, 3rd order is not visible.

$$n_{max} = 2.$$

Result : As $n_{max} = 2$.

Problem 8

A plane transmission grating having 6000 lines/cm is used to obtain a spectrum of light from a sodium lamp in the second order. Calculate the angular separation between the two sodium lines whose wavelengths are 5890 A° and 5896 A°.

(M.U. Dec. 2016) (7 m)

Solution :

Data : $a+b = \frac{1}{6000}$ cm, $n = 2$, $\lambda_1 = 5890$ A°, $\lambda_2 = 5896$ A°.

Formula : $(a+b)\sin\theta = n\lambda$

Calculation : $(a+b)\sin\theta_1 = n\lambda_1$

$$\theta_1 = \sin^{-1}\left(\frac{n\lambda_1}{a+b}\right)$$

$$\theta_1 = \sin^{-1}(2\times 5890\times 10^{-8}\times 6000)$$

$$\therefore \quad \theta_1 = \sin^{-1}(0.7068) = 44.97^\circ$$

$$\theta_2 = \sin^{-1}\left(\frac{n\lambda_2}{a+b}\right)$$

$$\theta_2 = \sin^{-1}(2\times 5896\times 10^{-8}\times 6000)$$

$$\therefore \quad \theta_2 = \sin^{-1}(0.7075) = 45.03^\circ$$

$$\therefore \quad \theta_2-\theta_1 = 45.03^\circ - 44.97^\circ = 0.06^\circ$$

Result : Angular separation $= 0.06^\circ$

Problem 9

A plane transmission grating having 6000 lines / cm is used to obtain a spectrum of light from a sodium lamp in the second order. Calculate the angular separation between the two sodium lines whose wavelengths are 5890 A° and 5896 A°.

(M.U. Dec. 2007) (7 m)

Solution :

Data : $a+b = \frac{1}{6000}$ cm, $n = 2$,

$\lambda_1 = 5890$ A° $= 5890\times 10^{-8}$ cm,

$\lambda_2 = 5896$ A° $= 5896\times 10^{-8}$ cm.

Formula : $(a+b)\sin\theta = n\lambda$, $\quad n = 1, 2, 3, \ldots$

Calculation :

$$(a+b)\sin\theta_1 = n\lambda_1 = \lambda_1$$

$$(a+b)\sin\theta_2 = n\lambda_2 = \lambda_2$$

$$\sin\theta_1 = \frac{\lambda_1}{a+b}$$

$$\therefore \quad \theta_1 = \sin^{-1}\left(\frac{\lambda_1}{a+b}\right)$$

$$= \sin^{-1}(5890\times 10^{-8}\times 6000) = 20.69^\circ$$

and

$$\sin\theta_2 = \frac{\lambda_2}{a+b}$$

$$\therefore \quad \theta_2 = \sin^{-1}\left(\frac{\lambda_2}{a+b}\right)$$

$$= \sin^{-1}(5896\times 10^{-8}\times 6000) = 20.71^\circ$$

$$\therefore \quad \theta_2-\theta_1 = 20.71^\circ - 20.69^\circ$$

$$= 0.02^\circ = 1.2'$$

Result : Angular separation $= 0.02^\circ = 1.2'$.

Problem 10

The visible spectrum ranges from 4000 A° to 7000 A°. Find the angular breadth of the first order visible spectrum produced by a plane grating having 6000 lines / cm when light is incident normally on the grating.

(M.U. Dec. 2015, 19) (5 m)

Solution :

Data : $\lambda_1 = 4000$ A° $= 4\times 10^{-5}$ cm,

$\lambda_2 = 7000$ A° $= 7\times 10^{-5}$ cm, $n = 1$,

$a+b = \frac{1}{6000 \text{ lines/cm}}$.

Formula : $(a+b)\sin\theta = n\lambda$, $\quad n = 1, 2, 3, \ldots$

Calculation :

$$(a+b)\sin\theta_1 = \lambda_1$$

$$\theta_1 = \sin^{-1}\left(\frac{\lambda_1}{a+b}\right)$$
$$= \sin^{-1}(4 \times 10^{-5} \times 6000)$$
$$= \sin^{-1}(0.24) = 13.88°$$

$$(a+b)\sin\theta_2 = \lambda_2$$
$$\theta_2 = \sin^{-1}\left(\frac{\lambda_2}{a+b}\right)$$
$$= \sin^{-1}(7 \times 10^{-5} \times 6000)$$
$$= \sin^{-1}(0.42) = 24.83°$$
$$\therefore \quad \theta_2 - \theta_1 = 24.83° - 13.88° = 10.95°$$

Result : Angular separation = 10.95°.

Problem 11

A diffraction grating used at normal incidence gives a line 5400 A° in certain ord superimposed on another line 4050 A° of the next higher order. If the angle of diffracti 30°, how many lines / cm are there on the grating? **(M.U. May 2010; Nov. 2012) (5**

Solution :

Data : $\lambda_1 = 5400$ A° $= 5400 \times 10^{-8}$ cm,

$\lambda_2 = 4050$ A° $= 4050 \times 10^{-8}$ cm, $\theta = 30°$.

Formula : $(a+b)\sin\theta = n\lambda$, $n = 1, 2, 3, ...$

Calculation :

$$\left.\begin{aligned}(a+b)\sin\theta &= n\lambda_1\\ (a+b)\sin\lambda &= (n+1)\lambda_2\end{aligned}\right\} \text{since } n \propto \frac{1}{\lambda} \text{ for constant } (a+b) \text{ and } \theta$$
$$n\lambda_1 = (n+1)\lambda_2$$
$$n = \frac{\lambda_2}{\lambda_1 - \lambda_2}$$
$$\therefore \quad n = \frac{4050 \times 10^{-8}}{(5400-4050) \times 10^{-8}} = 3$$
$$(a+b)\sin\theta = n\lambda_1$$
$$(a+b) = \frac{n\lambda_1}{\sin\theta} = \frac{3 \times 5400 \times 10^{-8}}{\sin 30°}$$

$$\therefore \quad (a+b) = 3.24 \times 10^{-4}$$
$$\text{No. of lines / cm} = \frac{1}{a+b} = 3086$$

Result : No. of lines / cm = 3086.

Problem 12

How many orders will be observed by a grating having 4000 lines / cm if it is illuminated by a light of wavelength in the range 5000 A° to 7500 A°.

(M.U. May 2009) (7 m)

Solution :

Data : $a + b = \frac{1}{4000}$ cm.

$\lambda_1 = 5000$ A° $= 5000 \times 10^{-8}$ cm,

$\lambda_2 = 7500$ A° $= 7500 \times 10^{-8}$ cm.

Formula : $(a+b)\sin\theta = n\lambda$, $n = 1, 2, 3, ...$

Calculations :

For $n = n_{max}$, $\sin\theta = 1$, $(a+b) = n_{max} \cdot \lambda$

For λ_1, $n_{max} = \frac{a+b}{\lambda_1} = \frac{1}{4000 \times 5000 \times 10^{-8}} = 5$

For λ_2, $n_{max} = \frac{a+b}{\lambda_2} = \frac{1}{4000 \times 7500 \times 10^{-8}} = 3.3 \approx 3$

Orders observed are from n = 3 to n = 5.

Result : For the wavelength range of 5000 A° to 7500 A°, three orders will be observed - the 3rd, the 4th and the 5th orders.

Problem 13

In an experiment with grating, third order spectral line of some wavelength coincides with the fourth order spectral line of wavelength 4992 A°. Calculate the value of the wavelength.

(M.U. Dec. 2011) (7 m)

Solution :

Data : Let the unknown wavelength be λ_1.

$n_1 = 3$ for λ_1, $n_2 = 4$ for $\lambda_2 = 4992$ A°.

$\lambda_2 = 4992 \times 10^{-8}$ cm

Formula : $(a+b)\sin\theta = n\lambda, \quad n = 1, 2, 3, ...$

Calculations :

$$(a+b)\sin\theta = n_1\lambda_1$$
$$(a+b)\sin\theta = n_2\lambda_2$$
$$n_1\lambda_1 = n_2\lambda_2$$
$$\therefore \quad \lambda_1 = \frac{n_2\lambda_2}{n_1} = \frac{4\times 4992\times 10^{-8}}{3}$$
$$= 6656\times 10^{-8}\ \text{cm.}$$

Result : Unknown wavelength = 6656 A°.

Problem 14

A diffraction grating used at normal incidence gives a yellow line ($\lambda = 6000$ A°) [in a] certain spectral order superimposed on a blue line ($\lambda = 4800$ A°) of next higher order. I[f the] angle of diffraction is $\sin^{-1}(3/4)$ calculate the grating element.

(M.U. Dec. 2015; May 2016, 17)

Solution :

Data : $\lambda_1 = 6000\ \text{A°} = 6\times 10^{-5}$ cm.

$\lambda_2 = 4800\ \text{A°} = 4.8\times 10^{-5}$ cm.

$\theta = \sin^{-1}(3/4)$.

Formula : $(a+b)\sin\theta = n\lambda, \quad n = 1, 2, 3, ...$

Calculation : For given (a + b) and θ, $n \propto \frac{1}{\lambda}$.

$$\therefore \quad (a+b)\sin\theta = n\lambda_1$$
$$\therefore \quad (a+b)\sin\theta = (n+1)\lambda_2$$
$$n\lambda_1 = (n+1)\lambda_2$$
$$\therefore \quad n = \frac{\lambda_2}{\lambda_1-\lambda_2} = \frac{4.8\times 10^{-5}}{(6-4.8)\times 10^{-5}} = 4$$
$$a+b = \frac{n\lambda_1}{\sin\theta} = \frac{4\times 6\times 10^{-5}}{(3/4)} = 32\times 10^{-5}\ \text{cm.}$$

Result : Grating element = 32×10^{-5} cm.

Problem 15

Calculate the maximum order of diffraction maxima seen from plane transmission grating with 2500 lines per inch if light of wavelength 6900 A° falls normally on it.

(M.U. Dec. 2017) (5 m)

Solution :

Data : $N = \frac{1}{a+b} = 2500\ \text{lines / inch} = 2500\times 2.52\times 10^{-2} = 63\ \text{lines / m}$

$\lambda = 6900\ \text{A°} = 6900\times 10^{-10}$ m.

Formula : $(a+b)\sin\theta = n\lambda$

Calculation : For $n = n_{max}$, $\sin\theta = 1$

$$\therefore \quad n_{max} = \frac{a+b}{\lambda} = 2.3$$

Result : $n_{max} = 2$.

Problem 16

A plane transmission grating has 5000 lines / cm.

(a) Determine the highest order of spectrum observed if incident light has $\lambda = 6000$ A°.

(b) If the opaque spaces between the slits are exactly double the transparent space and the maximum order observed is three, find which order of spectra will be absent.

Solution :

Data : (i) $\lambda = 6000\ \text{A°} = 6000\times 10^{-8}$ cm., $a+b = \frac{1}{5000}$ cm

(ii) $a = 2b$, $\quad n_{max} = 3$.

Formula : $(a+b)\sin\theta = n\lambda, \quad n = 1, 2, 3, ...$

Calculation :

(i) $(a+b)\sin\theta_{max} = n_{max}\cdot\lambda$

$$n_{max} = \frac{a+b}{\lambda} = \frac{1}{5000\times 6000\times 10^{-8}}$$
$$\therefore \quad n_{max} = 3.33 \approx 3$$

(ii) $(a+b)\sin\theta = n\lambda$: maxima

$b\sin\theta = n\lambda$: minima

$$\frac{a+b}{b} = \frac{m}{n}$$

$$\therefore \quad m = \frac{a+b}{b}n = \frac{2b+b}{b}n = 3n$$

$$m = 3, 6, 9, ...$$

Since $n_{max} = 3$, absent order = 3 only.

Result : (i) Maximum visible order, $n_{max} = 3$

(ii) Absent order, $m_{max} = 3$.

Problem 17

A grating has 620 rulings/mm and is 0.5 mm wide. What is the smallest waveleng[th] interval that can be resolved in the third order at $\lambda = 481$ nm? **(M.U. May 2016, 17)**

Solution :

Data : $N = 620 \times 0.5 = 310$, $\lambda = 481 \times 10^{-9}$ m, $m = 3$.

Formula : $$\frac{\lambda}{d\lambda} = mN$$

Calculations : $$d\lambda = \frac{\lambda}{mN} = \frac{481\times10^{-9}}{3\times310} = 0.5172 \times 10^{-9} \text{ m}$$

Result : $d\lambda = 0.5172$ A°

Problem 18

Light is incident on a grating of 0.5 cm width with 3000 lines. (i) Find angu[lar] separation in 2nd order of two sodium lines 5890 A° and 5896 A°. (ii) Check whether th[e] two lines are resolved in 2nd order or not. **(M.U. Dec. 2010) (6**

Solution :

Data : Width of grating = 0.5 cm, n = 2,

Total No. of lines on the grating = 3000,

$\lambda_1 = 5890$ A° $= 5890 \times 10^{-8}$ cm,

$\lambda_2 = 5896$ A° $= 5896 \times 10^{-8}$ cm.

Formula : $(a+b)\sin\theta = n\lambda$, $n = 1, 2, 3, ...$

$$\text{R.P.} = mN = \frac{\lambda}{d\lambda}$$

Calculations :

(i) No. of lines / cm on the grating $= \frac{3000}{0.5} = 6000$

$$a+b = \frac{1}{6000} = 1.667 \times 10^{-4} \text{ cm}$$

$$(a+b)\sin\theta_1 = 2\lambda_1$$

$$\theta_1 = \sin^{-1}\left(\frac{2\lambda_1}{a+b}\right) = \sin^{-1}\left(\frac{2\times5890\times10^{-8}}{1.667\times10^{-4}}\right)$$

$$\therefore \quad \theta_1 = 44.96°$$

$$(a+b)\sin\theta_2 = 2\lambda_2$$

$$\theta_2 = \sin^{-1}\left(\frac{2\lambda_1}{a+b}\right) = \sin^{-1}\left(\frac{2\times5896\times10^{-8}}{1.667\times10^{-4}}\right)$$

$$\therefore \quad \theta_1 = 45.03°$$

$$\theta_2 - \theta_1 = 45.03° - 44.96° = 0.07°$$

(ii) $$\lambda = \frac{\lambda_1+\lambda_2}{2} = 5893 \times 10^{-5} \text{ cm}$$

$$d\lambda = \lambda_2 - \lambda_1 = 6 \times 10^{-5} \text{ cm.}$$

$$\text{R.P.} = \frac{\lambda}{d\lambda} = \frac{5893\times10^{-5}}{6\times10^{-5}} = 997.166$$

The number of lines on the grating 1 cm required to resolve λ_1 and λ_2 is

$$N = \frac{\lambda}{m\,d\lambda} = \frac{997.166}{2} = 498.38$$

$$\therefore \quad N = 498$$

The no. of lines / cm on the given grating is $= \frac{3000}{0.5} = 6000$ which is quite large.

Hence, the sodium lines will be well resolved.

Result : Both the wavelengths 5890 A° and 5896 A° will be well resolved in the 2nd order.

Problem 19

Which particular spectra would be absent when the width of the opacity is do that of the transparency in a grating? (M.U. Dec. 2010)

Solution :

Data : $a = 2b$

Formula : $(a+b)\sin\theta = m\lambda$: maxima

$b \sin\theta = n\lambda$: minima

$$m = \frac{a+b}{b} n.$$

Calculations : $m = \frac{2b+b}{b} n = 3n$

$m = 3, 6, 9, ...$

Result : The 3^{rd}, 6^{th}, 9^{th} orders will remain absent.

Problem 20

Calculate the minimum number of lines required on a grating that can just reso the two sodium lines $\lambda_1 = 5890$ A° and $\lambda_2 = 5893$ A° in the third order. (M.U. Dec. 2014) (5

Solution :

Data : $\lambda_1 = 5890\text{ A°} = 5890 \times 10^{-8}$ cm,

$\lambda_2 = 5893\text{ A°} = 5893 \times 10^{-8}$ cm, $m = 3$.

Formula : Resolving power $= \frac{\lambda}{d\lambda} = mN$.

Calculations : $\lambda = \frac{\lambda_1+\lambda_2}{2} = \frac{(5890+5896)\times 10^{-8}}{2}$

$= 5893 \times 10^{-8}$ cm

$d\lambda = (5896 - 5890) \times 10^{-8} = 6 \times 10^{-8}$ cm

$$N = \frac{\lambda}{m\,d\lambda} = \frac{5893 \times 10^{-8}}{3 \times 6 \times 10^{-8}} = 327.38$$

Hence, the grating surface needs minimum of 327.38 lines on it which means 3 lines, at least.

Result : Minimum of 328 lines are required.

Problem 21

Deduce the missing orders for a double slit Fraunhoffer diffraction pattern if the slit widths are 0.16 mm and they are 0.8 mm apart.

Solution :

Data : $b = 0.16$ mm $= 0.016$ cm, $a = 0.8$ mm $= 0.08$ cm

Formula : Missing orders, $m = \left(\frac{a+b}{b}\right) n$

Calculations : $m = \frac{0.08 + 0.016}{0.016} n = 6n$

$m = 6, 12, 18, ...$

Result : The 6^{th}, 12^{th}, 18^{th}, ... etc. orders will be absent.

Problem 22

Find the maximum resolving power of a grating 2 cm wide with 6000 lines / cm illuminated by a light of wavelength 5890 A°.

Solution :

Data : Width of grating surface = 2 cm

$a + b = \frac{1}{6000}$ cm, $\lambda = 5890\text{ A°} = 5890 \times 10^{-8}$ cm.

Formula : $RP = mN$, $(a+b)\sin\theta = m\lambda$, $m = 1, 2, 3, ...$

Calculations :

$(RP)_{max} = m_{max} \cdot N$

$m_{max} = \frac{a+b}{\lambda}$, since $\theta_{max} = 1$

$\therefore$ $m_{max} = \frac{1}{6000 \times 5890 \times 10^{-8}} = 2.8$

Here, $2 < m_{max} < 3$.

Hence, $m_{max} = 2$.

$\therefore$ $N = 6000$ lines / cm × 2 cm = 12000 lines

$RP_{max} = 2 \times 12000 = 24000$

Rsult : Maximum resolving power = 24000.

Problem 23

What is the highest order spectrum which can be seen with monochromatic lig wavelength 6000 A° by means of a diffraction grating with 5000 lines / cm.

(M.U. Dec. 2013; May 2017)

Solution :

Data : $\lambda = 6000\ A° = 6 \times 10^{7}$ m, No. of lines / cm = 5000

Formula : $(a + b) \sin\theta = n\lambda$; $n = 1, 2, 3, ...$

Calculations : $(a+b) = \frac{1}{5000} = 2 \times 10^{-6}$ m

For $n = n_{max}$, $\sin\theta = 1$

$$n_{max} = \frac{a+b}{\lambda}$$

$$n_{max} = \frac{2\times10^{-6}}{6\times10^{-7}} = 3.3$$

Result : Highest order = 3.

Problem 24

Calculate the maximum order of diffraction maxima seen from a plane diffrac grating having 5500 lines/cm if light of wavelength 5896 A° falls normally on it.

(M.U. May 2015) (5

Solution :

Data : No. of lines / cm = 5500, $\lambda = 5896\ A° = 5896 \times 10^{-7}$ m.

Formula : $(a + b) \sin\theta = n\lambda$; $n = 1, 2, 3, ...$

Calculations : For $n = n_{max}$, $\sin\theta = 1$

$$n_{max} = \frac{a+b}{\lambda}$$

$$a+b = \frac{1}{5500} = 1.818 \times 10^{-6}\ m$$

$$n_{max} = \frac{1.818\times10^{-6}}{5896\times10^{-10}} = 3.08$$

Result : Maximum order, $n_{max} = 3$.

Problem 25

A diffraction grating used at normal incidence gives a yellow line ($\lambda = 6000$ A°) in a certain spectral order superimposed on a blue line ($\lambda = 4800$ A°) of next higher order. If the angle of diffraction is $\sin^{-1}$ (3/4), calculate the grating element. **(M.U. Dec. 2015) (5 m)**

Solution :

Data : $\lambda_1 = 6000\ A° = 6 \times 10^{-7}$ m, $\lambda_2 = 4800\ A° = 4.8 \times 10^{-7}$ m.

$\theta = \sin^{-1}(3/4)$ ∴ $\sin\theta = (3/4) = 0.75$.

Formula : $(a + b) \sin\theta = n\lambda$; $n = 1, 2, 3, ...$

Calculations : For given (a + b) and θ,

$$n \propto \frac{1}{\lambda}$$

Since $\lambda_1 > \lambda_2$,

$$(a+b)\sin\theta = n\lambda_1$$

$$(a+b)\sin\theta = (n+1)\lambda_2$$

$$\therefore \quad n\lambda_1 = (n+1)\lambda_2$$

$$\therefore \quad n = \frac{\lambda_2}{\lambda_1 - \lambda_2} = \frac{4.8\times10^{-7}}{6\times10^{-7} - 4.8\times10^{-7}} = 4$$

$$a+b = \frac{n\lambda_1}{\sin\theta} = \frac{4\times6\times10^{-7}}{0.75} = 32\times10^{-7}\ m$$

Result : Grating element = 3.2×10^{-6} m.

Problem 26

A plane grating just resolves two lines in the second order. Calculate the grating element if $d\lambda = 6$ A°, $\lambda = 6 \times 10^{-5}$ cm and the width of the ruled surface is 2 cm.

(M.U. Dec. 2013) (5 m)

Solution :

Data : $d\lambda = 6\ A° = 6 \times 10^{-10}$ m, $m = 2$, $\lambda = 6 \times 10^{-5}$ cm $= 6 \times 10^{-7}$ m.

Width of ruled surface = 2 cm.

Formula : $\frac{\lambda}{d\lambda} = mN$, $a + b = \frac{1}{\text{Number of lines/cm}}$

Calculations : $N = \frac{\lambda}{m d\lambda} = \frac{6 \times 10^{-7}}{2 \times 6 \times 10^{-10}} = 500$

$$\text{No. of lines/cm} = \frac{N}{2} = \frac{500}{2} = 250$$

$$\text{Grating element, } (a+b) = \frac{1}{250} = 4 \times 10^{-3} \text{ cm}$$

Result : Grating element $= 4 \times 10^{-5}$ m.

Problem 27

Calculate the minimum number of lines in a grating which will just resolve in first order the wavelengths 5890 A° and 5896 A°. **(M.U. Dec. 2014)** (5

Solution :

Data : $\lambda_1 = 5890 \text{ A}° = 5890 \times 10^{-10}$ m,

$\lambda_2 = 5896 \text{ A}° = 5896 \times 10^{-10}$ m, $m = 1$.

Formula : $\frac{\lambda}{d\lambda} = mN$

Calculations : $d\lambda = 6 \text{ A}° = 6 \times 10^{-10}$ m

$$\lambda = \frac{\lambda_1 + \lambda_2}{2} = 5893 \text{ A}° = 5893 \times 10^{-10} \text{ m}$$

$$N = \frac{\lambda}{m d\lambda} = \frac{5893 \times 10^{-10}}{1 \times 6 \times 10^{-10}} = 982.166$$

Result : Minimum number of lines required = 982.

Problem 28

Light is incident normally on a grating 0.5 cm wide with 2500 lines. Find the angular separation of the two sodium lines in the first order spectrum. Can they be seen distinctively if the lines are 5890 A° and 5896 A°?

Solution :

Data : Number of lines/cm = 5000, m = 1,

$\lambda_1 = 5890 \text{ A}°$, $\lambda_2 = 5890 \text{ A}°$.

Formula : $(a+b) \sin\theta = n\lambda$, $mN = \frac{\lambda}{d\lambda}$

Calculations :

$$(a+b)\sin\theta_1 = \lambda_1$$

$$\therefore \quad \theta_1 = \sin^{-1}\left(\frac{\lambda_1}{a+b}\right)$$

$$= \sin^{-1}(5890 \times 10^{-8} \times 5000)$$

$$\therefore \quad \theta_1 = 17.1275°$$

$$(a+b)\sin\theta_2 = \lambda_2$$

$$\therefore \quad \theta_2 = \sin^{-1}\left(\frac{\lambda_2}{a+b}\right)$$

$$= \sin^{-1}(5896 \times 10^{-8} \times 5000)$$

$$\therefore \quad \theta_2 = 17.1455°$$

$$\therefore \quad \theta_2 - \theta_1 = 17.1455° - 17.1275° = 0.0177°$$

Here, $d\lambda = 6 \text{ A}°$, $\lambda = 5893 \text{ A}°$

$$N = \frac{\lambda}{m d\lambda} = \frac{5893 \times 10^{-8}}{6 \times 10^{-8}} = 982$$

Number of lines / cm required on the grating = 982.

Number of lines / cm available on the grating = 50000

Results : $\theta_2 - \theta_1 = 0.0177°$

These lines will be resolved well by the grating.

Important Points to Remember

1. Fraunhoffer Diffraction

   (i) At single slit, $I = I_0 \frac{\sin^2\beta}{\beta^2}$

   (ii) At N slit, $I = I_0 \frac{\sin^2\beta}{\beta^2} \cdot \frac{\sin^2 N\gamma}{\sin^2\gamma}$

   where $\beta = \frac{\pi b \sin\theta}{\lambda}$ and $\gamma = \frac{\pi d \sin\theta}{\lambda}$.

2. Diffraction maxima : $(a+b)\sin\theta = n\lambda$

   Diffraction minima : $b \sin\theta = n\lambda$

3. $(a+b)$ = grating element

4. $a+b = \dfrac{1}{\text{No. of lines / cm on the granting surface}}$

5. a = width of opaque space (rulings), b = slit width.

6. Absent spectra : $m = \left(\dfrac{a+b}{b}\right) n$

m = order of absent maxima,

n = order of regular minima (1, 2, ...)

7. Maximum order : $m_{max} = \dfrac{a+b}{\lambda}$

8. Resolving power of a grating : $RP = \dfrac{\lambda}{d\lambda} = mN$.

$\lambda = \dfrac{\lambda_1 + \lambda_2}{2}$, $d\lambda = (\lambda_1 - \lambda_2)$

EXERCISE

(A) Short Answer Type Questions

1. What are the types of diffraction? Differentiate between them.
2. What are called absent spectra? Explain.
3. Define a diffraction grating. What is grating element?
4. What is called the maximum visible order in a diffraction spectrum? Explain.
5. What are the advantages of increasing the number of rulings on the grating?
6. Explain Rayleigh's criteria for resolution.
7. Explain how is the number of missing orders dependent on the dimensions of grating.

(B) Long Answer Type Questions

1. Discuss the phenomenon of Fraunhoffer diffraction at a single slit and obtain conditions for maxima and minima.

3. Discuss the phenomenon of Fraunhoffer diffraction at N slit and obtain the conditi for maxima and minima.

4. What is a diffraction grating and the grating element? Explain the experimental method of determination of wavelength of spectral line using diffraction grating.

5. State Rayleigh's criteria for resolution. Define resolving power of an optical instrument. Derive an expression for the resolving power of a plane transmission grating.

(C) Problems for practice

1. Red light of wavelength 7500 A° is normally incident on a plane transmission grating with 6000 lines / cm. How many diffraction orders are observed? [Ans. : 2]

2. A diffraction grating with 3086 lines / cm gives a line (5400 A°) in a certain order superposed on another line of the next higher order. If the angle of diffraction is 30°, calculate the wavelength of the second line. [Ans. : 4050 A°]

3. In a grating with 5000 line / cm for is wavelength of 6000 A°, what is the highest order spectrum observed? If $a = 2b$ which order of spectra will be absent? [Ans. : 3, 3rd, 6th, 9th, ...]

4. The light of wavelength 6000 A° is incident normally on a plane diffraction grating of 1000 lines / cm. Calculate :

(i) The difference between the wavelengths that just appear separated in the first order, and

(ii) The resolving power of the third order spectrum. [Ans. : 6 A°, 3000]

5. Light is incident normally on a grating 0.5 cm wide with 2500 lines. Find the angle of diffraction for the principal maximum of the two sodium lines the first order spectrum [$\lambda_1 = 5890$ A° and $\lambda_2 = 5896$ A°]. Are these two lines resolved? [Ans. : Yes]

Previous University Examination Questions with Solutions

1. Define diffraction of light. Why is it not evident in daily life?
[Refer § 1.1] **(M.U. May 2008) (3 m)**

2. What do you mean by diffraction? State its types and differentiate between them.
[Refer § 1.2.4] **(M.U. May 09, 11; Dec. 2009, 11, 15) (3 m)**

3. Discuss the phenomenon of Fraunhoffer's diffraction at a single slit and obta[in] condition for the 1^{st} minimum. (M.U. May 2000?) (
[Refer § 1.3.1]

4. For plane transmission grating, prove that the condition for diffraction maxi[ma] $d \sin \theta = n \lambda$, $n = 1, 2, 3, ...$ (M.U. May 2018; Dec. 2014, 19) (
[Refer § 1.3.2]

5. Describe the construction of a diffraction grating. What is grating and grating elem[ent] Explain the experimental method of determination of wavelength of a spectral using diffraction grating. (M.U. May 2008, 11, 13, 16, 17; Dec. 2009, 12, 17) (
[Refer § 1.4, 1.4.1 (A), 1.6]

6. What is diffraction grating? What is the advantage of increasing the number of l[ines] in a grating? (M.U. May 2010; Dec. 2011, 14) (
[Refer § 1.4 and 1.5.2]

7. What is grating element ? Explain how the number of lines on grating decide maximum number of orders of diffraction.
[Refer § 1.4.1 (A) and 1.52] (M.U. May 2012, 14; Dec. 2013, 16) (

8. What is diffraction grating and grating element? Explain the experimental me[thod] to determine the wavelength of a spectral line using a diffraction grating.
[Refer § 1.4, 1.4.1 (A) and 1.6] (M.U. Nov. 2018) (7

9. *What is Rayleigh's criteria of resolution? Write the expression for the resolv[ing] power of a grating.* (M.U. May 2010, 11, 13, 14, 15; Dec. 2016, 17) (3
[Refer § 1.5.2]

10. What is absent spectra? Derive the condition for absent spectra in grating.
[Refer § 1.4.1 (B)] (M.U. May 2013, 18; Nov. 2018; Dec. 2010, 16, 19) (3

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MODULE 2

Laser and Fibre Optics

(Prerequisites : Absorption, recombination, energy bands of p-n junction, Refractive index of a material, Snell's law.)

Laser : Spontaneous emission and stimulated emission, Metastable state, Population inversion, Types of pumping, Resonant cavity, Einsteins's equations, Helium Neon laser, Nd:YAG laser, Semiconductor laser, Applications of laser- Holography.

Fibre optics : Numerical Aperture for step index fibre, Critical angle, Angle of acceptance, V number, Number of modes of propagation, Types of optical fibres, Fibre optic communication system. **(06 Hours)**

(Weightage - 27%)

Course Outcome : CO2 : Learner will be able to illustrate the working principle of various Lasers and their applications in different fields, the concept of optical fibre and its applications in communication system.

SYNOPSIS

2.1 Introduction of LASER
2.2 Quantum processes of Laser Production : Absorption, Spontaneous emission and Stimulated emission.
2.3 Einstein's Coefficients
2.4 Basic Requirements of LASER Production : Population inversion, Pumping, Metastable state, Resonant Cavity.
2.5 Types of Pumping
2.6 LASER Sources : He-Ne, Nd-YAR, Semiconductor
2.7 Application of Laser : Holography
Important Points to Remeber
Exercise
Previous University Examination Questions with Solutions

2.8 Introduction of Fibre Optics
2.9 Principle of Fibre Optics
2.10 Basic construction and Types of Optical Fibres
2.11 Numerical Aperature of a Step Index Fibre, Critical Angle and Angle of Acceptance
2.12 Modes of Propagation : V number
2.13 Applications of Optical Fibres : Fibre Optic Communication Sytem
2.14 Solved Problems
Important Points to Remeber
Exercise
Previous University Examination Questions with Solutions

LASER

2.1 Introduction

LASER is acronym for Light Amplification by Stimulated Emission of Radiation.

LASER is a mechanism to produce a light beam with very special features. LASER light has the following characteristics :

(i) High monochromaticity, (ii) High coherence.
(iii) High directionality, (iv) High intensity, and
(v) Low divergence,
(vi) On the otherhand, ordinary light is polychromatic, incoherent and highly divergent.

Due to these characteristics LASER radiations are used in communication systems, reading and recording of data, defense technology, holography and so on.

2.2 Quantum Processes of Laser Production

2.2.1 : Principle

- The principle of LASER production is based on the theory of interaction of radiation with matter.
- A material medium is composed of identical atoms or molecules each of which is characterized by a set of discrete energy levels. The atoms can transit between any pair of energy levels when they receive or release an amount of energy equal to the energy difference between the two states.
- For simplicity, consider a substance in which atoms have only two allowed energy states, the ground state, E_1 and the excited state, E_2 with $E_2 - E_1 = h\nu$, the energy of a photon.
- When this substance is exposed to a radiation of a stream of photons each carrying energy, hν, three distinct interaction process can take place. These are absorption spontaneous emission and stimulated emission as shown in Figs. 2.1, 2.2 and 2.3 respectively.

(i) **Absorption :** When an atom in the ground state E_1 absorb an incident photon its energy increases by an amount hν and it goes to the excited state E_2. This process is called ***absorption*** that can be represented by

$$A + h\nu \rightarrow A^*$$

where, A = ground state photon

and A^* = excited state photon.

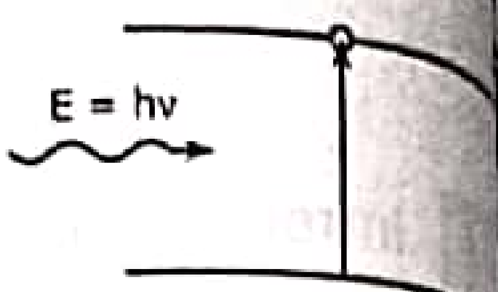


Fig. 2.1 : Absorption

(ii) **Spontaneous emission :** Normally the excited state is an unstable state where the lifetime of an atom is very short, around 10^{-8} sec. Hence the atom in the excited state, E_2 returns to the ground state spontaneously by releasing one photon of energy, hv. This process is called ***spontaneous emission*** which can be represented by

$$A^* \rightarrow A + h\nu$$

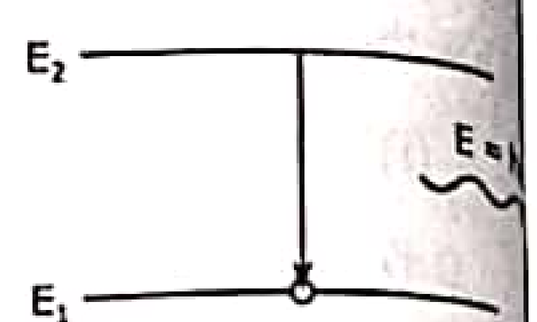


Fig. 2.2
Spontaneous emission

(iii) **Stimulated emission :** In this process an incident photon is absorbed by an excited atom as a result of which the atom becomes unstable in the state E_2 and makes a transition to the ground state releasing two photons. This process is called ***stimulated emission*** which can be written as

$$A^* + h\nu \rightarrow A + 2h\nu$$

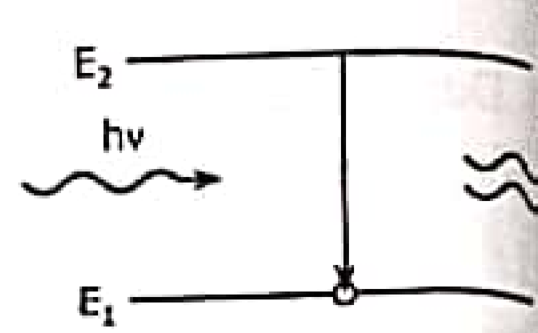


Fig. 2.3
Stimulated emission

2.2.2 : Amplification by Stimulated Emission of Radiation

The emitted photons propagate in the same direction as that of the incident pho[ton] or the stimulating photon. These two photons are identical in all respects to the stimulat[ing]

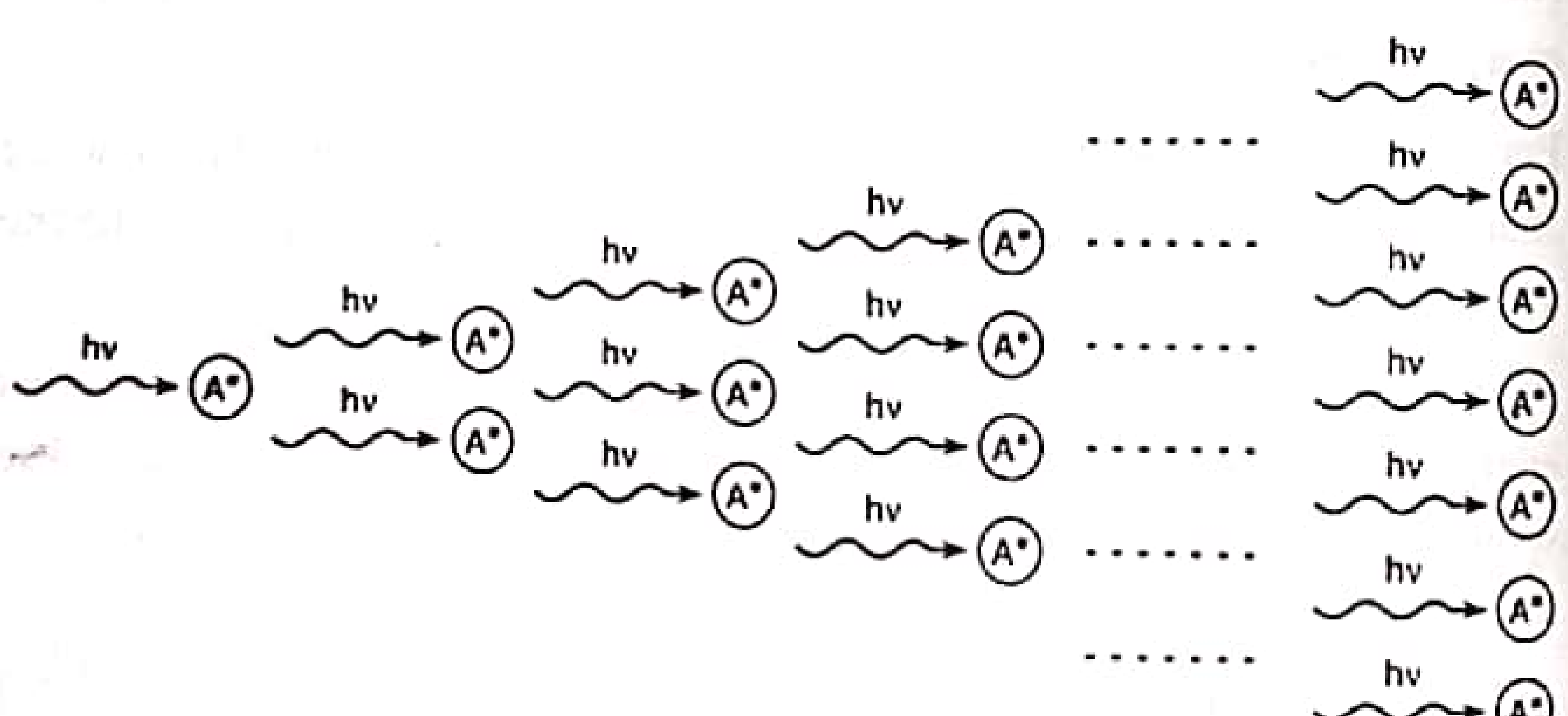


Fig. 2.4 : Amplification

photon. Hence, these two photons will stimulate two more excited atoms resulting in four resultant photons. These four photons in turn stimulate four more excited atoms and generate eight photons, and so on. This is shown in Fig. 2.4.

- The number of photons is built up in an avalanche manner.
- As all the light waves are generated from one initial wave all of them are coherent. Being in phase they interfere constructively the net intensity of which becomes

$$I_{total} \propto N^2$$

where N is the number of atoms present in the material.

Since the number of atoms in the material medium is very large, coherent emission leads to an enormously high intense light.

2.2.3 : Comparison of Stimulated Emission and Spontaneous Emission

Table 2.1

Spontaneous emission	Stimulated emission
1. It is a random and uncontrolled process.	1. It is a controlled process.
2. Being uncontrolled it can take place between any pair of energy states. Hence. Photons can carry any amount of energy.	2. It is possible only between two specified energy states. Hence, all emitted photons carry equal amount of energy.
3. As the transition take place randomly the emitted radiation are not in phase and also directed in all possible directions.	3. All the emitted photons are in phase and travel in the same direction.
4. Interference of emitted radiations take place constructively and destructively also. Hence the intensity of the emitted radiation becomes moderate.	4. The emitted radiations being in phase interfere constructively only. Hence the intensity becomes very high.
5. In this process photon multiplication does not take place. Hence there is no amplification of light.	5. Light amplification occurs due to multiplication of photons.

2.3 Einstein's Coefficients

- Consider a medium with atoms having only two allowed energy states E_1 a... with population N_1 and N_2 respectively. When a radiation of density 'ρ' is in... on this medium, the three interaction processes, absorption, spontaneous emi... and stimulated emission take place with probabilities P_{abs}, $P_{sp.em}$ and $P_{st.em}$. T... shown in Fig. 2.5.
- The probability of the absorption process to occur depends on :

  (i) The number of atoms present at the ground state (N_1) and

  (ii) The photon density (ρ) of the incident radiation.

  Hence, it is given by

$$P_{abs} = A_{12}\,\rho\,N_1 \quad \text{.......... (2.1-a)}$$

where A_{12} is the proportionality constant called the ***Einstein's coefficient of absorption.***

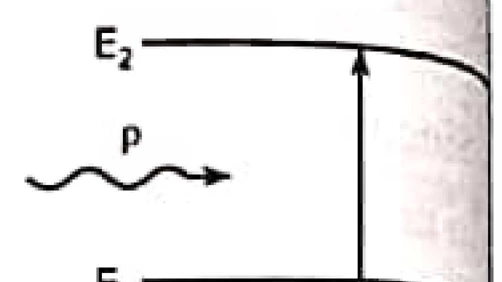


Fig. 2.5 (a)

- The probability that the spontaneous emission will take place depends on the number of atoms are excited to the higher state (N_2) and is given by

$$P_{sp.\,em} = B_{21}\,N_2 \quad \text{.......... (2.1-b)}$$

where B_{12} is the proportionality constant called the ***Einstein's coefficient of spontaneous emission.***

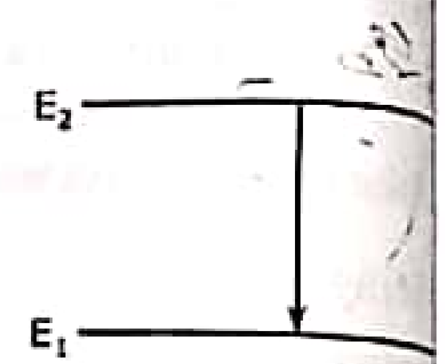


Fig. 2.5 (b)

- Lastly, the stimulated emission takes place with a probability that depends on :

  (i) The number of atoms present at the excited state (N_2) and

  (ii) The photon density of the incident radiation (ρ) and is given by

$$P_{st.\,em} = C_{21}\,\rho\,N_2 \quad \text{.......... (2.1-c)}$$

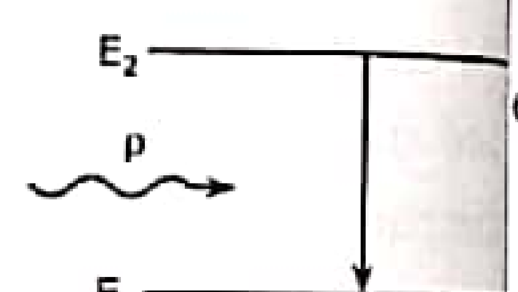


Fig. 2.5 (c)

where the proportionality constant C_{21} is called the ***Einstein's coefficient of stimul... emission.***

- In thermal equilibrium, the probability of transition from state E_1 to E_2 mus... equal to the same from E_2 to E_1. Thus,

$$P_{abs} = P_{sp.\,em} + P_{st.\,em} \quad \text{..........(}$$

$$A_{12}\,\rho\,N_1 = B_{21}\,N_2 + C_{21}\,\rho\,N_2$$

$$\rho\,(A_{12}\,N_1 - C_{21}\,N_2) = B_{21}\,N_2$$

$$\therefore \quad \rho = \frac{B_{21}\,N_2}{A_{12}\,N_1 - C_{21}\,N_2} = \frac{B_{21}\,N_2 / C_{21}\,N_2}{\frac{A_{12}}{C_{21}}\,\frac{N_1}{N_2} - 1}$$

$$= \frac{B_{21}/C_{21}}{\frac{A_{12}}{C_{21}} \times e^{(E_2 - E_1)/kT} - 1}$$

Here $N_1 = e^{-E_1/kT}$ and $N_2 = e^{-E_2/kT}$ by Maxwell - Boltzmann distribution where, 'k' is Boltzmann constant and 'T' is the absolute temperature.

$$\therefore \quad \rho = \frac{B_{21}/C_{21}}{\frac{A_{12}}{C_{21}} \times e^{h\nu/kT} - 1} \quad \text{.................... (2.3)}$$

Comparing this with Planck's radiation formula,

$$\rho = \frac{8\pi h \nu^3}{c^3}\left(\frac{1}{e^{h\nu/kT} - 1}\right) \quad \text{.................... (2.4)}$$

It is found that

$$\frac{B_{21}}{C_{21}} = \frac{8\pi h \nu^3}{c^3} \quad \text{.................... (2.5)}$$

and $$\frac{A_{12}}{C_{21}} = 1 \qquad i.e., \qquad A_{12} = C_{21} \quad \text{.................... (2.6)}$$

Equation (2.6) shows that the Einstein's coefficients for absorption and stimulated emission are equal. This can be explained as follows :

Taking ratio of equation (2.1-c) and (2.1-a), it is found that

$$\frac{P_{st.em.}}{P_{abs}} = \frac{C_{21}\,\rho\,N_2}{A_{12}\,\rho\,N_1}$$

Using equation (2.6) in this, the ratio becomes

$$\frac{P_{st.em.}}{P_{abs}} = \frac{N_2}{N_1} = e^{-(E_2 - E_1)/kT} = e^{-h\nu/kT} \quad \text{.................... (2.7)}$$

For ordinary visible light with an average wavelength of λ = 5000 A° at room temperature T = 300 K

$$\frac{P_{st.em.}}{P_{abs}} = e^{-hc/\lambda kT} = 10^{-44} \quad \text{......(}$$

which is negligible.

This means that stimulated emission does not occur naturally. It needs to be ind artificially to produce LASER.

2.4 Basic Requirements for LASER Production

When a radiation is incident on a material medium, all the three processe absorption, spontaneous emission and stimulated emission take place. Of these t processes stimulated emission is essential for the production of LASER. Hence, stimul emission should dominate absorption and spontaneous emission. To make it possible requirements are as follows :

2.4.1 Population Inversion

- This is a state of matter in which the number of atoms in the excited sta higher than that in the ground state.
- This can be explained by considering equations (2.1-a) and (2.1-c). For LAS production it requires to be

$$P_{st.\,em.} > P_{abs}$$

$$C_{21}\,\rho\,N_2 > A_{12}\,\rho\,N_1$$

Here $C_{21} = A_{12}$, as seen in equation (2.27). Hence,

$$N_2 > N_1 \quad \text{......(}$$

The chances of stimulated emission taking place increases when the stat population inversion is achieved in the medium.

2.4.2 : Pumping

- Usually atoms have a tendency to return to the ground state releasing absorbed energy. Hence, the population of the ground state is found to be gre than that of the higher excited state.
- Thus, the state of population inversion can not be achieved naturally. It ha be induced artificially by continuously raising a large number of atoms to higher energy state with continuous supply of external energy. This is ca the pumping mechanism.

Various methods of pumping that are generally used are as follows :

(i) **Optical pumping :** In this case optical energy is incident on the atomic system. On absorbing photons of the required energy value the atoms transit to the higher energy states.

(ii) **Electrical pumping :** A strong electric field is applied to the atomic system with the use of a high voltage power supply. The high energy electrons collide with the atoms and transfer their kinetic energy to the later. As a result atoms rise to the higher energy state.

(iii) **Direct conversion :** In this method the electrical energy directly creates the state of population inversion and LASER is produced. Here the electrical energy is directly converted into optical energy.

2.4.3 : Metastable State : Need of a Three Level System

- In the excited states atoms have a very short life time of around 10^{-8} sec. hence the excited atoms have a natural tendency to rapidly de-excite to the ground state through spontaneous emission.
- Even though atoms are continuously pumped to the excited state it is not possible to achieve the state of population inversion.
- Atoms do not stay in the excited state for a significant period of time so as to be stimulated by the incident radiation.
- Therefore, a third kind of energy state called the metastable state is required. In a metastable state the life time of atoms is around 10^{-3} sec. which is much longer than the time required for spontaneous emission to take place. Hence, a large number of atoms get accumulated at this level making the population inversion and stimulated emission possible.
- Metastable states are not artificially created. These are naturally present in between the ground state and the excited state in some materials as shown in Fig. 2.6.

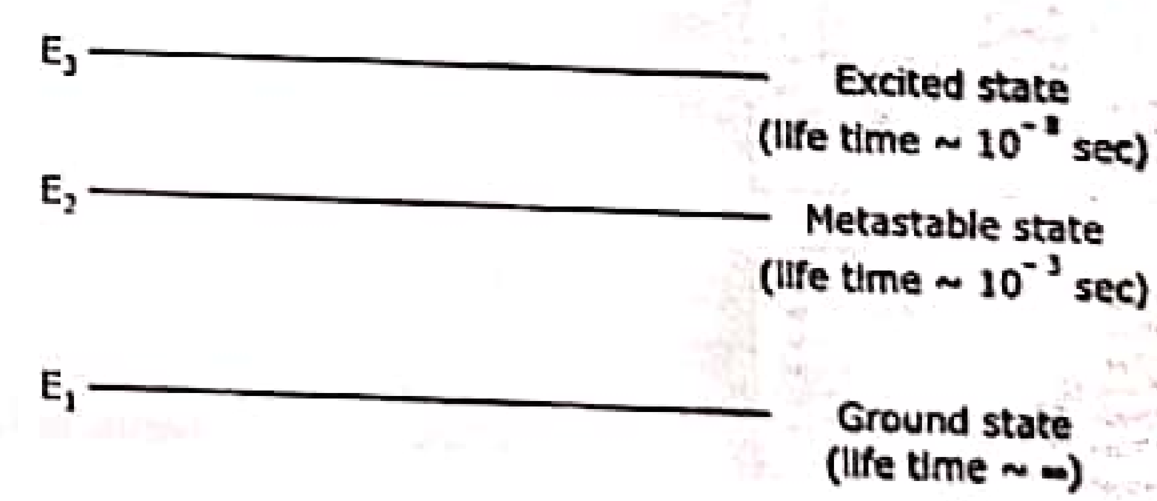


Fig. 2.6 : A three level atomic system

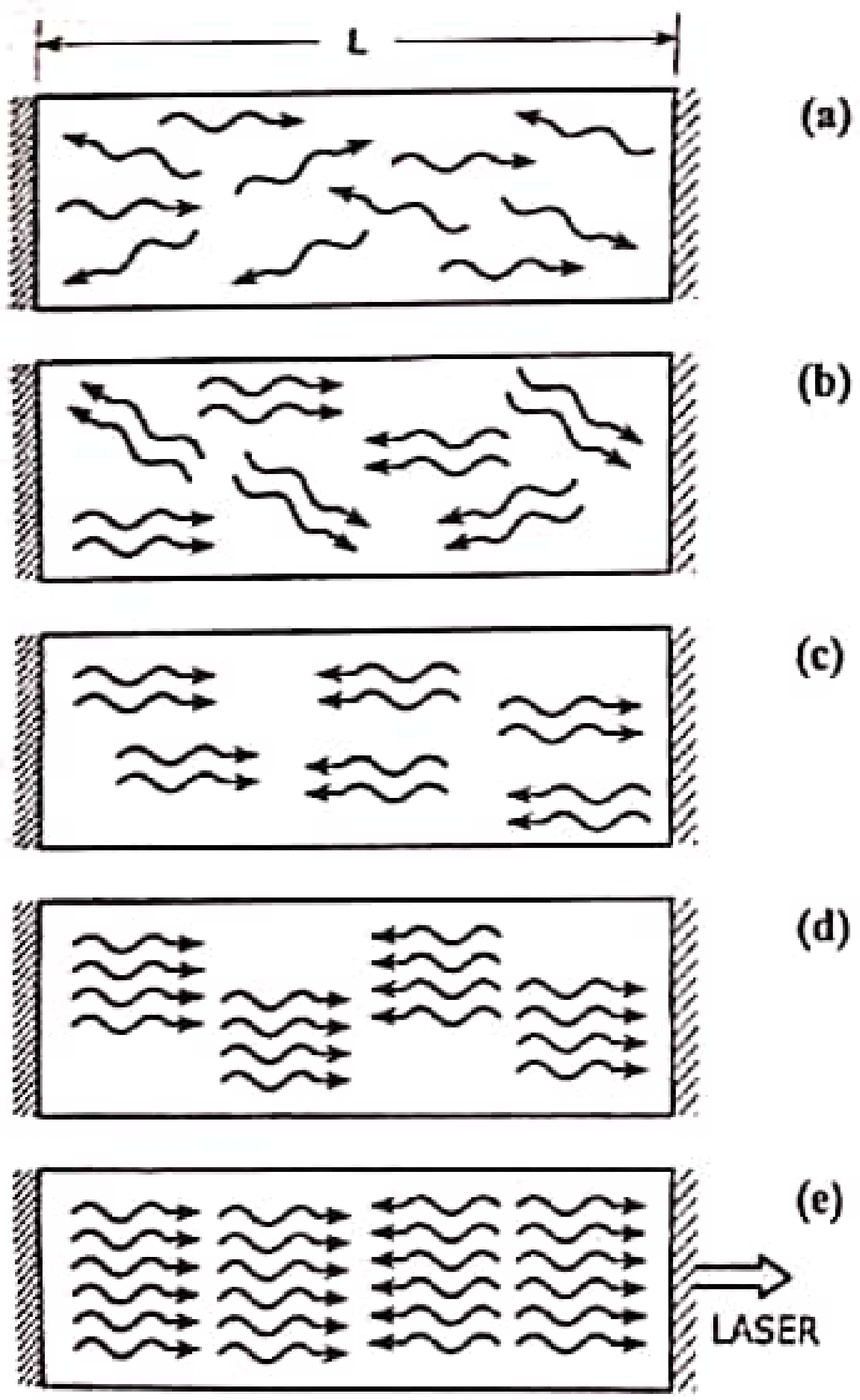
- Therefore, for LASER production *the materials to be used need to h… least three allowed atomic energy levels: the ground state, the meta… state and the excited state.*

2.4.4 : Resonant Cavity : Optical Resonator

A pair of optically flat parallel mirrors one totally reflecting and the other pa… reflecting constitutes the optical resonator or resonant cavity. The two reflectors e… the medium that emit LASER radiation.

The functions of a resonant cavity are explained in Fig. 2.8 as follows :

1. Light initially generated in a LASER source is primarily due to sponta… emission. These spontaneous photons being incoherent in nature d… contribute to the LASER beam. The photons, instead of being wasted, c… utilized as the trigger photons for the induction of stimulated emission. I… a resonant cavity the spontaneous photons are fed back to the medium f… purpose.



(a) Immediately after the atoms are pumped into excited states spontaneous photons are emitted.

(b) Spontaneous photons initiate stimulated emission and the pairs of coherent photons are released in all directions.

(c) Light waves traveling along the axis form standing waves ($L = n\lambda / 2$) between the reflectors by reflections and other waves are lost.

(d) Each of the photons in its back and forth journey triggers more stimulated emission and create more coherent photons.

(e) The large number of coherent waves interferes constructively and high amplitude, high intensity LASER beam comes out of the partial reflector.

Fig. 2.7

2. The stimulated photons, which are coherent in nature, when undergo multiplication by repetitive reflections at the two reflectors interfere constructively and a higher amplified LASER beam is produced.

2.5 Types of Pumping : Three Level and Four Level Lasing Schemes

Generally there are two types of pumping schemes used in LASER production. These are :

a) Three Level Pumping Scheme

The three energy levels involved in the atomic transition for laser production are the ground state, E_1, one metastable state, E_2 and the excited state, E_3.

The principle of LASER production in three levels pumping is explained in Fig. 2.8.

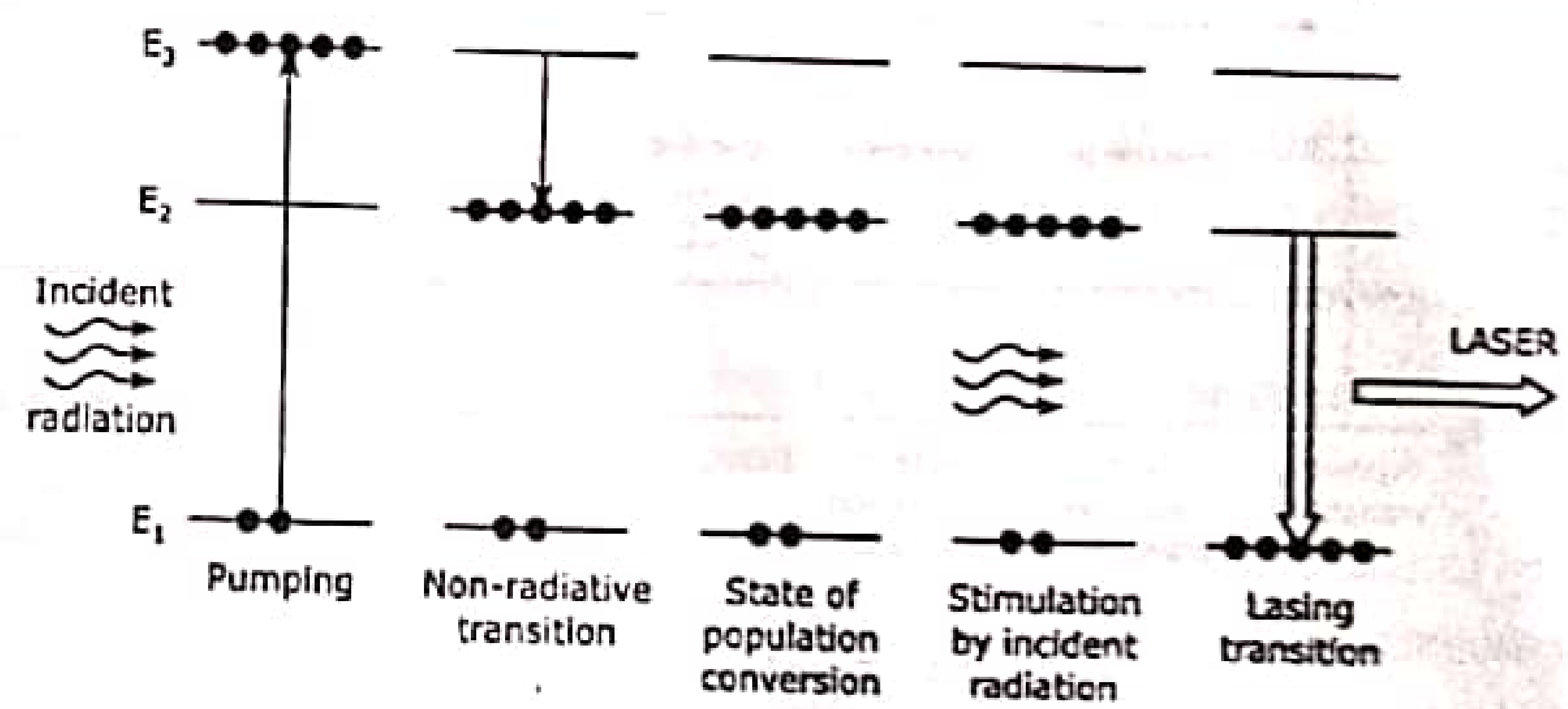


Fig. 2.8 : Three level Laser

- The atoms are raised to the excited state by pumping transition.
- Having a very short life time in the excited state, atoms immediately goes down to the metastable state by non radiative transition, emitting some heat energy.
- Since spontaneous emission is not possible in a metastable state atoms stay in level E_2 for a significant period of time. This results into population inversion between states E_2 and E_1.
- The medium keeps on receiving the incident radiation. The incident radiation initiates the atoms of level E_2 to transit simultaneously to the ground state. The lasing transition occurs between levels E_2 and E_1.

- In this scheme population inversion occurs between the metastable sta and ground state E_1 which is possible only when more than half of the gr state atoms are pumped to the higher state. This requires a very high pum power.
- Once the lasing transition is over the metastable state is empty and the gr state is full. The next lasing is possible only after the population invers re-established. Thus, the three level laser operates in pulsed mode.

(b) Four Level Pumping Scheme

- In this case the atoms of the material used have four allowed energy le the ground state, E_1, two metastable states, E_2 and E_3 and the excited stat
- The Lasing mechanism in a four level system is explained in Fig. 2.9.

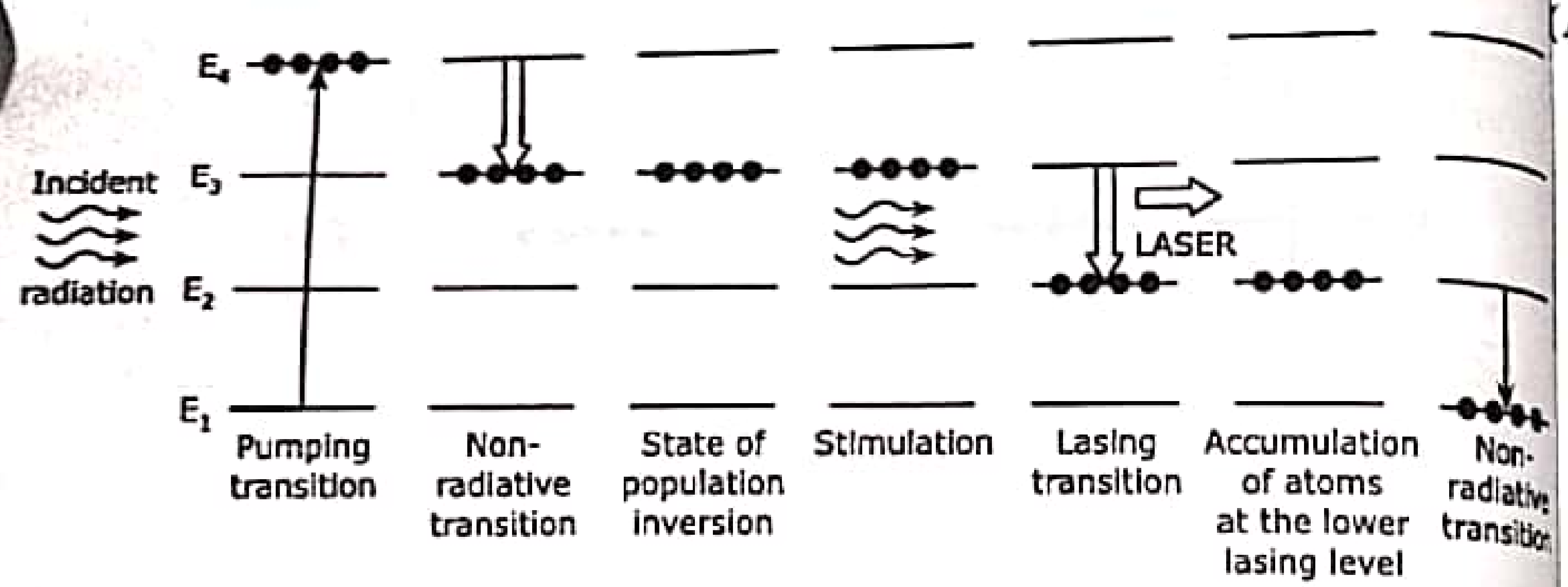


Fig. 2.9 : Four level Laser

- By pumping transition atoms are raised to the excited state, E_4.
- Being unstable in the excited state, E_4 almost immediately the atoms und non radiative transition emitting heat energy and reach the upper metas state E_3.
- Population inversion occurs between the metastable states E_3 and E_2.
- Being initiated by the incident energy all the atoms simultaneously tran the lower metastable state, E_2. Here E_3 is the upper lasing level and E_2 lower lasing level between which the lasing transition takes place and LA is emitted.
- All the atoms are accumulated in level E_2. As E_2 is a metastable state can not go down to the ground state by spontaneous emission. Instead b radiation transition the atoms to the ground state.

- By the time the first lot of atoms returns to the ground state another lot of atoms through pumping reaches the level E_3 re-establishing population inversion between levels E_3 and E_2 which eventually results in Laser production. Thus a four level laser operates in continuous wave mode.
- As the lower lasing level E_2 is almost vacant very small pumping power is required to achieve population inversion between the upper Lasing level E_3 and the lower lasing level E_2.

2.6 LASER Sources : He-Ne, Nd-YAG, Semiconductor

2.6.1 : Fundamentals of a LASER Source

(A) The basic components of a LASER source :

These are as follows :

(i) **An active medium :** This is a material in which the state of population inversion can be induced easily. The wavelength of the emitted LASER beam depends on the energy values of the lasing levels of the active medium.

(ii) **An energy source :** This is required for pumping the atoms from ground state to higher energy levels.

(iii) **A resonant / optical cavity :** This is required for the feedback of spontaneous photons to the active medium and the multiplication of the stimulated photons which is essential for LASER production.

B) The mechanism of a LASER source :

This involves

(i) Excitation,

(ii) Population inversion, and

(iii) Cavity response.

C) Classification of LASER Sources

LASERs are classified according to the active medium used in the source, as follows :

(i) Solid LASER

(ii) Liquid LASER

(iii) Gas LASER

(iv) Dye of chemical LASER, and

(v) Semiconductor LASER

A few important LASER sources are described below.

2.6.2 : He-Ne Laser : A Four Level Gaseous Source Laser

The essential components of this source are as follows :

(i) **Active medium :** This is a mixture of Helium and Neon gases with He : = 10 : 1 ratio, filled in a pyrex tube of diameter about 1.5 cm and leng about 35 cm. The pressure inside the tube is maintained at about 1 mm of Here 'He' is the *host gas* and 'Ne' is called the ***activator***, because 'Ne' ta part actively in lasing transition.

(ii) **Energy source :** This is a high voltage power source of about 4 kV connected to the pyrex tube to excite the active medium. Hence, the pum is electrical pumping.

(iii) ***Resonant / optical cavity :*** A pair of reflectors, one total, and one partial fi *to the two inner end surfaces of the pyrex tube.*

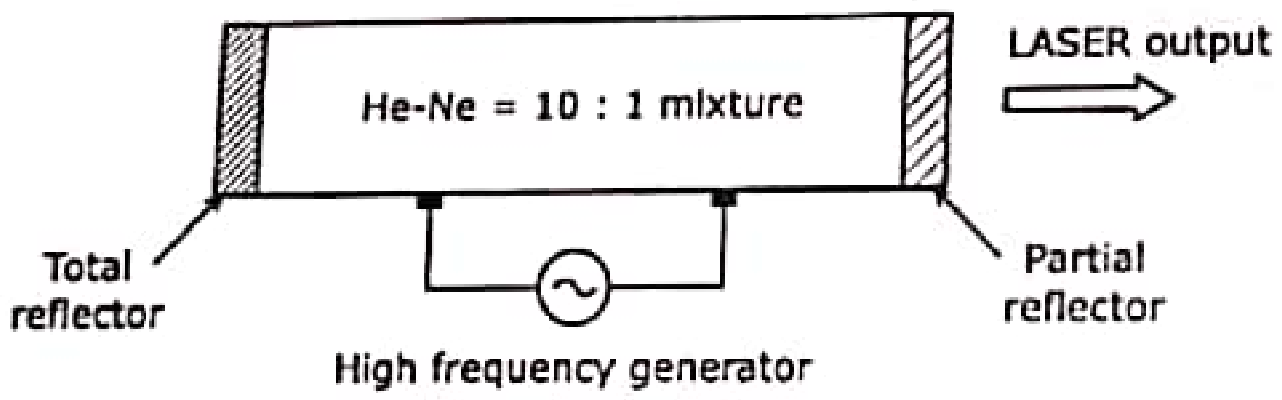


Fig. 2.10 : Construction of He-Ne laser.

Working of He - Ne Laser

The working of the Laser source can be explained with the help of its energy le diagram shown in Fig. 2.11.

- As the high voltage power supply is switched on, the high energy electr start flowing through the gas mixture and collide with He and Ne atoms the time of impact the electrons transfer their kinetic energy to the gas ato He atoms being fairly lighter than Ne atoms, absorb the electron kinetic ene and are excited to the metastable levels F_2 and F_3 easily from the ground st F_1. This is the pumping transition of He atoms. During this transistion, the atoms exist in the ground state.

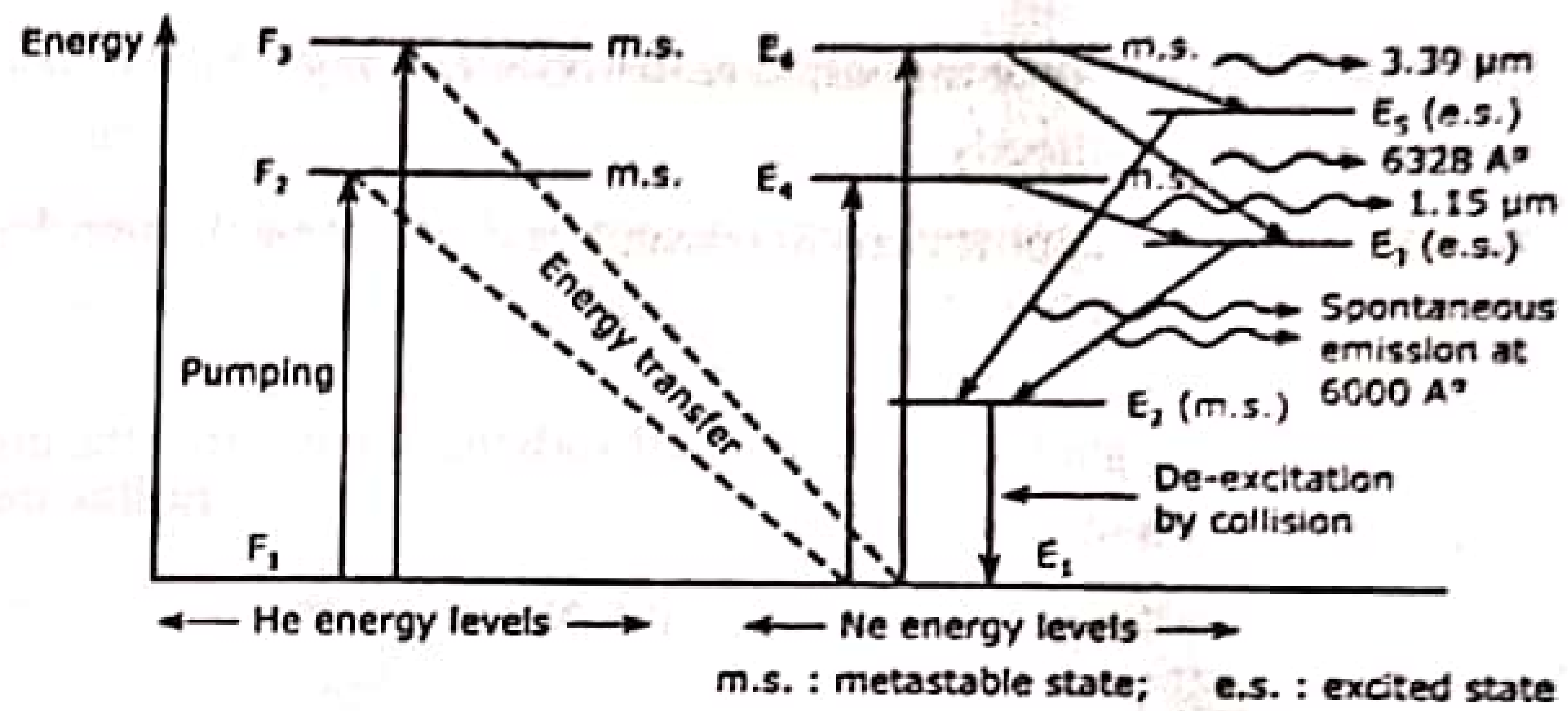


Fig. 2.11 : Energy level diagram of He - Ne LASER

- The two gases are in close energy level structures as shown in Fig. 2.28. The excited He atoms collide with the ground state Ne atoms and transfer their energy to the later. As a result, the ground state Ne atoms are excited to the metastable energy states E_6 and E_4 which are almost parallel to the levels F_3 and F_2 of He respectively. The He atoms return to the ground state.
- This causes population inversion in Ne at E_6 and E_4 levels with respect to E_5 and E_3 levels and lasing occurs on three possible transitions as :

  $E_6 \rightarrow E_5$ emitting LASER of wavelength **3.39 μm** in the **infrared** region,

  $E_6 \rightarrow E_3$ emitting LASER of wavelength **6328 A°** in the **visible** region,

  and $E_4 \rightarrow E_3$ emitting LASER of wavelength **1.15 μm** in the **infrared** region.
- The levels E_5 and E_3 are excited states where the life time of Ne atoms is about 10^{-8} sec. Hence, the Ne atoms transit to the metastable state E_2 from E_5 and E_3 levels by spontaneous emission.
- The energy level E_2 is metastable from which spontaneous emission is rare. Hence, Ne atoms are accumulated at level E_2. During their stay at E_2 level Ne atoms collide with the tube walls and give up their excess energy as heat energy and returns to the ground state.
- By the time the first set of Ne atoms return to the ground state one more set of Ne atoms are raised to level E_4 and E_6 inducing population inversion and resulting in Lasing action. Hence, He - Ne laser operates continuously and hence emits continuous wave.
- As one of the output is in the visible region (6328 A°) the output power is low ranging from 1 mW to 50 mW. Though the output power is low, due to coherent

radiation the intensity of the output radiation is very high. So it is dangero look at the source directly.

He - Ne Laser has applications in research and educational laboratories.

Role of He atoms

1. He atoms being lighter than Ne atoms absorb the energy from the high en electrons easily and very fast.
2. He atoms have longer lifetime than the Ne atoms at the metastable st Hence, the state of population inversion is maintained for a long time w makes the induction for stimulated emission easy.
3. The ratio He : Ne = 10 : 1 makes the probability of energy transfer fo atoms to Ne atoms much larger than that of the reverse.
4. Being a good conductor of heat He acts as a coolant and no separate coo system is required.

Merits

1. Continuous output Laser source
2. *Highly stable*
3. No separate cooling is required

Demerits

- Low efficiency and low power output.

2.6.3 : Nd - YAG Laser : A Four Level Solid Source LASER

Nd-YAG LASER is one of the most popular type of LASER.

(a) Components

The essential components of the source are as follows :

(i) **Active medium :** This is Yttrium Aluminium Garnet ($Y_3 Al_5 O_{12}$) or Y which is an optically isotropic crystal in which some of the Y^{3+} ions are repla by neodymium (Nd^{3+}) ions. Therefore, YAG act as the *host crystal*, and N ions are the ***activators*** which take part actively in lasing transition. The ac medium is in the form of a Nd - YAG rod typically of length 10 cm an diameter 12 mm.

(ii) **Energy source :** A krypton flash lamp in tube from is used as the energy source.

(iii) **Resonator / optical cavity :** The two ends of the Nd-YAG rod are polished and silvered partially at one side and totally at the other side to form the optical resonator.

(b) Construction

As seen in Fig. 2.12 the system consists of a pair of optically cylindrical reflectors housing the Nd-YAG rod along one focus line and the flash lamp along the other focus line. The light leaving one focus of the ellipse reaches the other focus after reflection from the inner silvered surface of the cylindrical reflectors. Thus the total flash lamp radiation is incident on the ND-YAG rod.

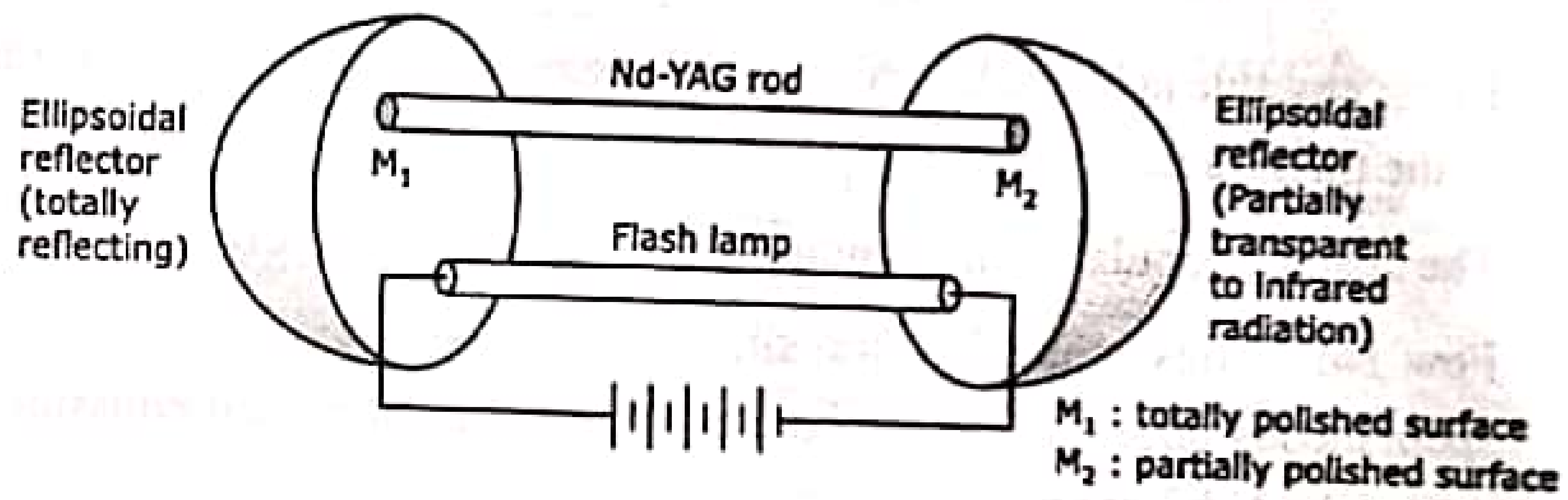


Fig. 2.12 : Nd-YAG LASER

Fig. 2.13 : Cross-sectional view of the reflections in the cylindrical reflector

(c) Working

- The working of the Nd-YAG LASER is explained with the help of the energy level diagram, given in Fig. 2.14.
- When krypton flash lamp is switched on it emits intense radiation of wavelength range 7000 A° to 8000 A°.

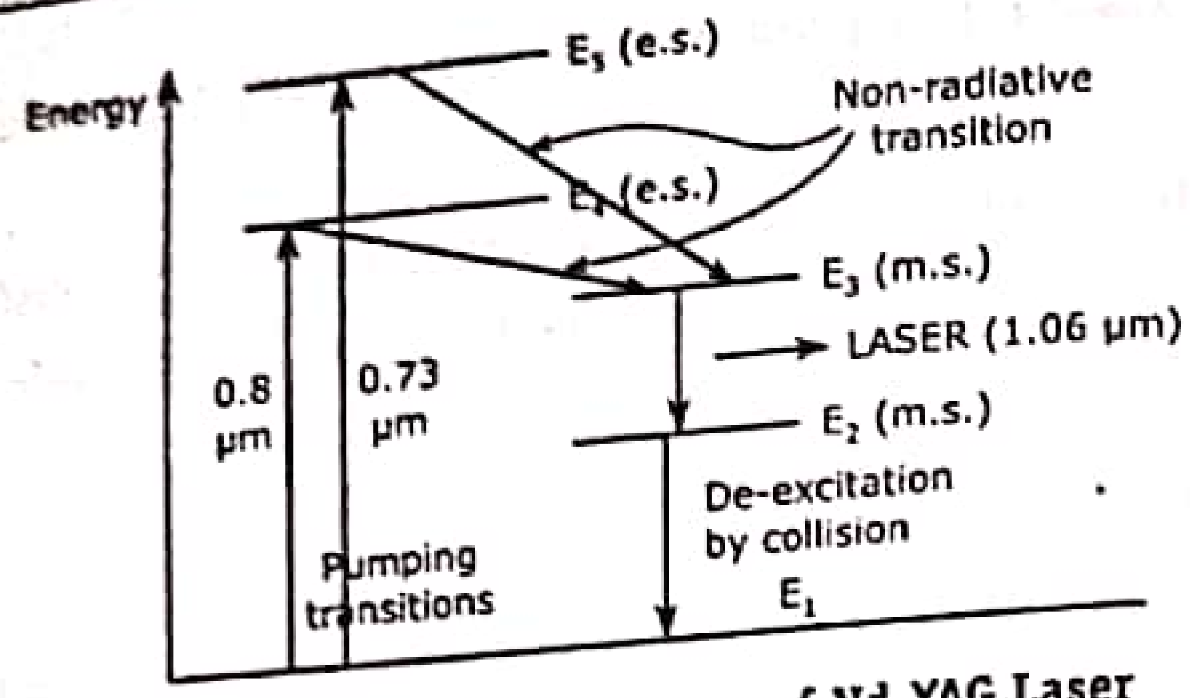


Fig. 2.14 : Energy level diagram of Nd-YAG Laser

- The Nd^{3+} atoms absorb this energy and are pumped to the excited sta[te] and E_5.
- Being unstable at levels E_4 and E_5, Nd^{3+} atoms make a non radiative tran[sition] to the metastable state E_3.
- The state of population inversion is developed between levels E_3 and E[2].
- Few Nd^{3+} ions make fast transition from level E_3 to level E_2 emi[tting] spontaneous photons. These photons initiate the stimulated emission [of] rest of the Nd^{3+} atoms.
- Lasing transition takes place from E_3 to E_2 level emitting LASER of wavel[ength] 10,600 A° in the infrared region.
- Being reflected back and forth between the refelectors, M_1 and M_2 [o]f resonant cavity the LASER beam becomes more intense and comes out [through] partial reflector M_2 and then through the ellipsoidal surface.

(d) Function of the Ellipsoidal Reflector

- In between the consecutive flashes of the krypton lamp the Nd-YAG rod [...] on receiving light through multiple reflections on the ellipsoidal reflect[or].
- The LASER beam is emitted continuously by ND-YAG rod. Hence, ND[-YAG] LASER works on the continuous wave (CW) mode.
- ND-YAG LASER has applications in welding and drilling in hardware ind[ustry], surgery, etc.

2.6.4 : Semiconductor LASER

(a) Principle

Semiconductor diode LASER is formed by a heavily doped forward biased p-n junction diode made up of compound semiconductors.

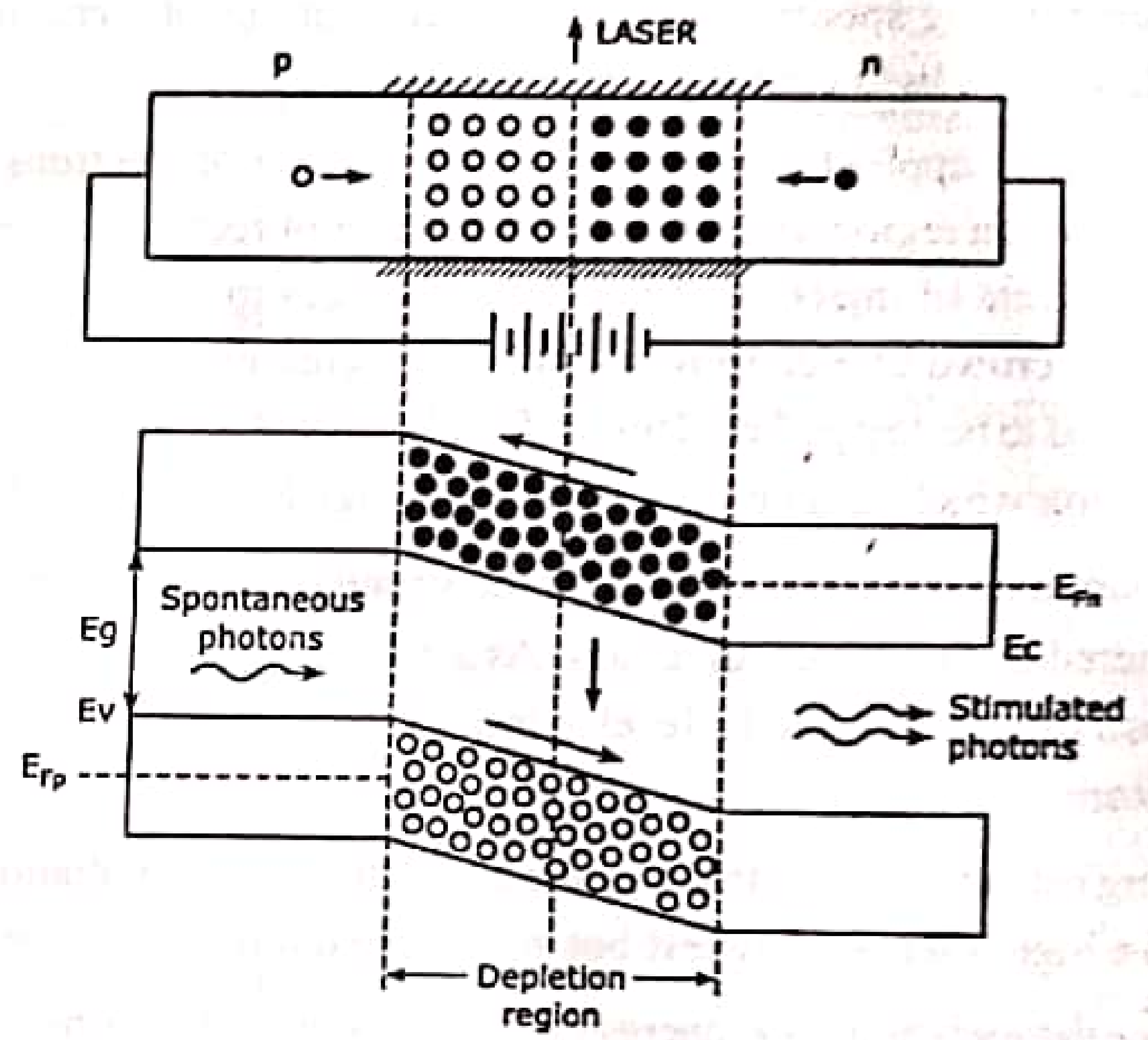


Fig. 2.15 : Semi conductor diode LASER with energy band diagram

The energy band diagram of this p-n junction is shown in Fig. 2.15. The diode has heavy doping of holes on the p side and that of electrons on the n side. Hence, the Fermi level, E_{Fp} on the p side enters the valence band and that on the n side, E_{Fn} enters the conduction band.

(b) Components

The essential components of this LASER source are as follows :

(i) **Active medium :** The depletion region of the diode act as the active medium, which ideally should be free of charge carriers.

(ii) **Energy source :** An electrical power supply is used as the energy source. In this case the pumping is of direct conversion type.

(iii) **Resonator cavity :** The two open sides of the depletion region are polished to serve as the partial and total reflectors. This serves as the resonator cavity with diode thickness equal to an integral multiple of $\lambda / 2$.

(c) Working

- The working of diode LASER depends on the forward current. In fo biasing, electrons and holes enter the depletion region.
- At low forward current these electrons and holes recombine in the dep region emitting spontaneous photons which are incoherent. In this mod diode act as a light emitting diode (LED).
- With a large applied forward bias a large number of electrons and holes the depletion region. A time comes when rate of recombination is much than the rate of injection of the charge carriers to the depletion region result, a crowd of electrons and holes is accumulated in this region wh supposed to be charge free. This artificially created state is the state of popu inversion which is developed between energy levels E_c and E_V.
- A spontaneous photon can trigger the stimulated emission of all the elec gathered in the depletion region. As a result all the electrons simultane transit from E_c level to E_v level to recombine with the holes releasing coh photons.
- Being reflected within the resonant cavity the emitted radiation is strength and a high intensity LASER beam is emitted through the partial reflecto
- Since the recombination energy is released as the LASER beam, the wavele of the LASER depends on the band gap energy, E_g of the dioide mat Hence,

$$E_g = h\nu = \frac{hc}{\lambda}$$

and the wavelength of the emitted LASER beam is given by

$$\lambda = \frac{hc}{E_g}$$

- For Ga As, the wavelength of the LASER is 8500 A° which is in the inf region.
- A GaAsP diode LASER operated at liquid nitrogen temperature emits LA of wavelength 6400 A° in the visible region.

(d) Merits and Demerits of Laser Diode

Merits :

1. It is simple and compact.
2. It is highly efficient.
3. It requires very low power.
4. It is tunable that means the wavelength of the emitted LASER can be regulated.

Demerits : It is highly temperature sensitive.

(d) Application of diode LASER

1. Used in optical fibre communications as the light source.
2. Used in satellite communications.
3. Used in LASER printers, copiers.
4. Used in CD players, optical floppy discs.

(e) Comparison of LED and Laser Diode

Table 2.2

Sr. No.	Parameters	LED	Laser diode
1.	Required forward bias	Low	High
2.	Light emission process	Spontaneous	Stimulated
3.	Population inversion	Not required	Essential
4.	Resonant cavity	Not required	Essential. Opposite sides of the pn junction are made parallel and reflecting to cause feedback
5.	Emitted radiation	Incoherent, low intensity, monochromatic	Coherent, high intensity, mono-chromatic.

2.7 Applications of LASER : Holography and Other Applications

2.7.1 Holography

This is a method of producing a three dimensional image of an object where... photography a two dimensional image is formed.

The technique of holography was discovered by Danis Gabor in 1948 but reache... full potential only after the discovery of LASER in 1960. This is because holog... requires a light wave which is highly coherent, highly monochromatic, highly intense, highly directional, the characteristics possessed by only a LASER radiation.

Photography

Photography is the technique of recording a two dimensional image of a t... dimensional object. The object is illuminated by the incident wave and the wave refle... from the object is called the object wave that carries the information about every poi... the object as shown in Fig. 2.16. Through the lens the object wave reaches the photogra... plate and a point to point (*e.g.*, A → A', B → B') recording of the object is done. Thus recorded image is identical to the object. Since the photographic plate is sensitive to inten... variations, it records the intensity of the object wave. The intensity (∝ amplitude²) is a... dimensional physical quantity. Hence, the photographic image is a two dimensional im... which cannot be viewed from different perspectives.

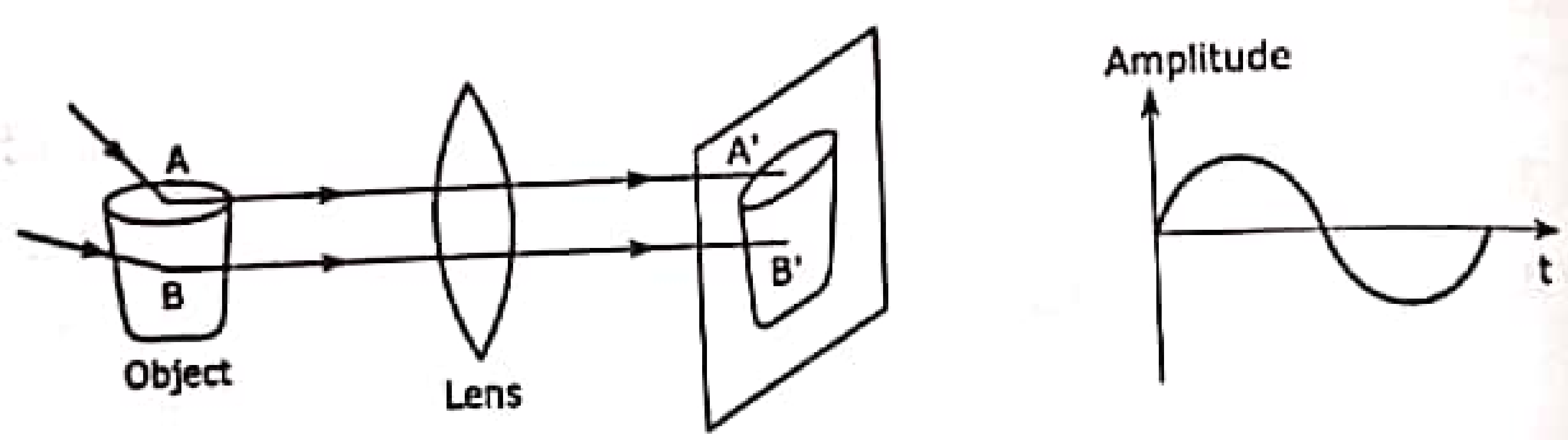


Fig. 2.16 : Photography

Holography

In holography, in addition to the two dimensions of intensity, a third dimen... the phase of light is also recorded. This is done by using the principle of interfere... The image produced by holographic technique has three dimensions and one... view the image from different perspectives.

The basic technique of holography involves two stages :

(i) Recording :

- A weak but broad beam of LASER is split into two beams, one is made to be incident on the object and the other on a mirror as shown in Fig. 2.18.
- The plane wave reflected from the object, the ***object wave*** which carries all the information about the object, reach the photographic plate. Simultaneously another plane wave reflected from the mirror, called the ***reference wave*** also strike the photographic plate.
- The object wave and the reference wave interfere on the recording plate and the intensity distribution is recorded on it.

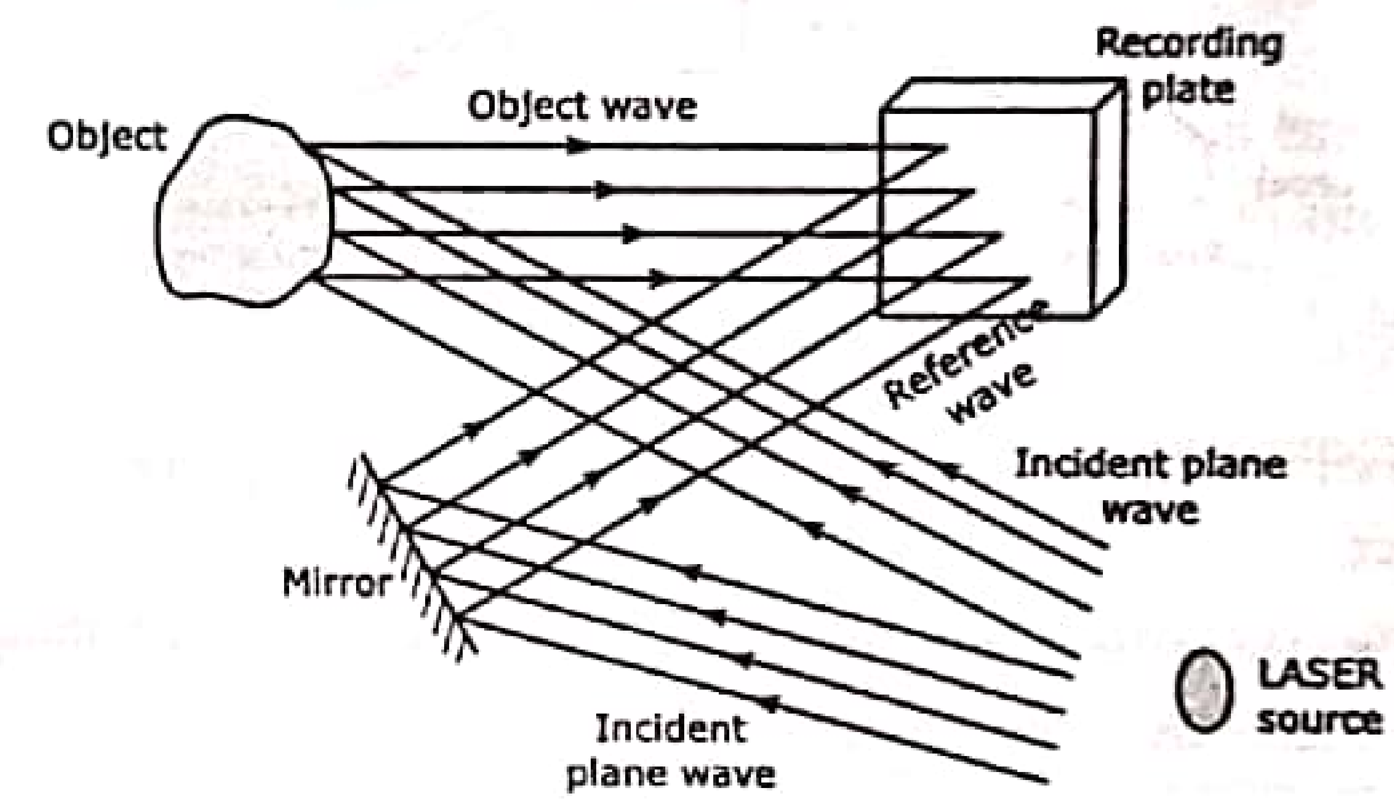


Fig. 2.17 : Recording of Hologram

- The recorded interference pattern is called the ***hologram*** which contains the information not only of the amplitude but also of the phase of the object wave.
- The hologram does not have any resemblance with the object but a photograph has. This is because the two interfering waves have very complex phase differences as the object wave is produced from different parts and depths of the object. Every point on the hologram contains the information of the entire object.
- The hologram represents a complex interference pattern with alternate dark and bright fringes in which the information of the object is optically coded.

(ii) Reconstruction :

- In this process a reconstruction wave identical with the reference wave is used to illuminate the hologram. If the hologram acts as a diffraction grating the process leads to a real and a virtual image of the object.

- The real image is formed in front of the hologram at the first order ma... one side if the observer is positioned at the other first order maxima. T... image can be recorded.
- The three dimensional vertical image is also formed behind the holog... shown in Fig. 2.18.

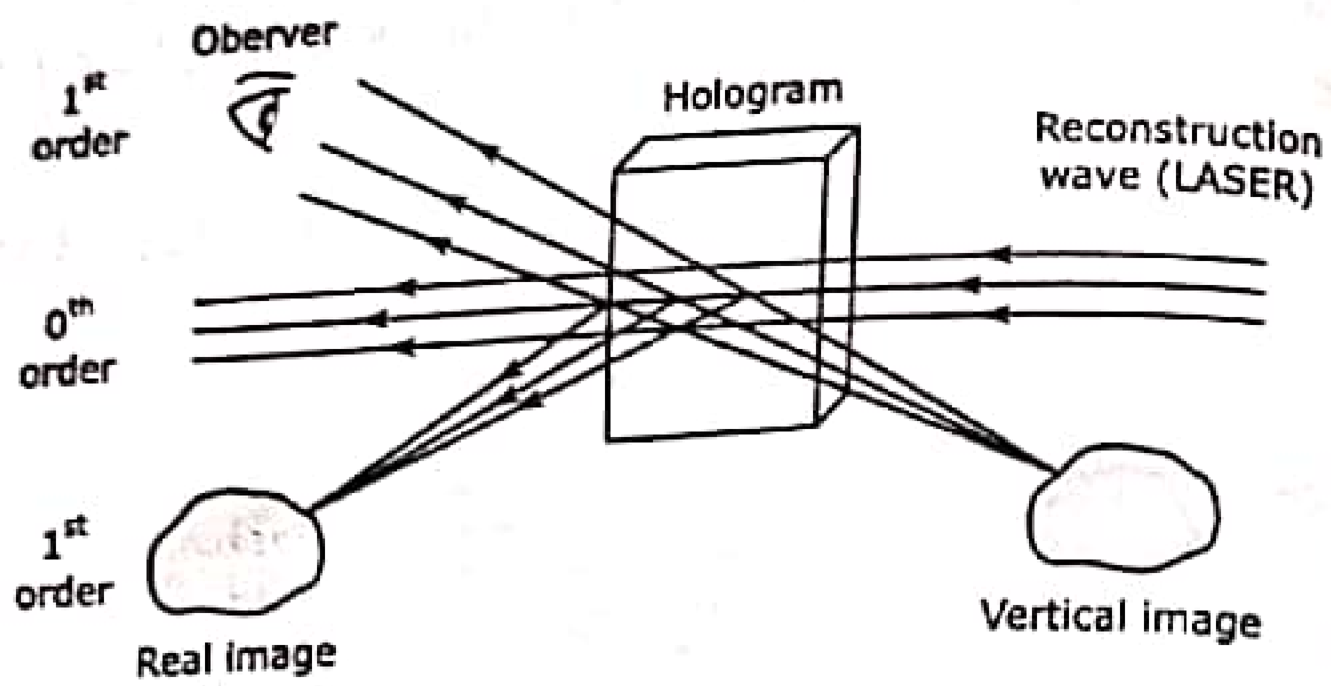


Fig. 2.18 : Reconstruction

- The virtual image can be viewed from different perspectives and it has... effect.
- The image is identical to the object. This fact is used to exhibit the va... like diamonds in the museum.

2.7.2 : Comparison of Holography with Photography

In photography, the intensity (2D) of the object wave is recorded on the photog... plate resulting in a two dimensional image (called the negative) of the three dimen... object.

In holography, in addition to the intensity (2D) the phase (1D) also of the inter... waves are recorded. This results into a three dimensional image (called the hologra... three dimensional object.

Table 2.3

Photograph (negative)	Hologram
1. Ordinary light can be used.	1. LASER beam is necessary.
2. Information of each point of the object is recorded at a correspondence point on the image. Hence there is a point to point correlation between the object and the image.	2. Information of the entire object is... at each point of the image.
3. From a torn or shattered photograph the image cannot be reconstructed.	3. The image can be reconstructed from even a small fragment of the hologram.

2.7.3 : Other Applications

(a) LASER Detection and Ranging (LIDAR)

This is an accurate distance measurement technique. The is used to measure the distance between the moon and the earth.

LIDAR technique has other applications also. It is used in atmospheric pollution monitoring system, surveying, detection of fog layers, altimeters in aircraft etc.

(b) Defence Applications

1. The Laser range finders (LRF) with Nd-YAG Laser source are used to locate the enemy tank and other targets.
2. Laser guided missiles are very effective weapons. Here a remote control device emitting an infrared Laser beam is used to guide the missile. Laser guns and Laser induced bombs are also used in defence.
3. Emergency action messages can be conveyed efficiently and quickly between submarines, satellites and aircrafts by Laser guided communications.

(c) Industrial Applications

1. Laser is used for welding, cutting, drilling, soldering, heat treatment etc.
2. Laser beam scanning is used in printing industry and in Laser printers.
3. Laser is used in memory and logic circuits in semiconductor chips in microelectronic industry.
4. A large amount of data can be stored in a compact disc (CD) using a Laser beam.
5. Lasers are used in bar code scanners in Library and supermarkets.
6. Lasers are used in optical fibre communications

(d) **Medical Applications**

1. Low intensity Lasers have therapeutic applications.
2. Laser radiation is efficient in hemorrhage control
3. Lasers are used in surgery
4. Lasers are used in cancer treatment.

(e) **Holographic Applications**

1. Using Laser beam huge data can be stored on a hologram.
2. Holograms are used on voter identity cards, credit cards, tickets, originals of software programs, certificates to prevent falsification.

Important Points to Remember

1. **LASER :** Light Amplification by Stimulated Emission of Radiation.
2. **Absorption :** $A + h\nu \rightarrow A^*$

   Spontaneous emission : $A^* \rightarrow A + h\nu$

   Stimulated emission : $A^* + h\nu \rightarrow A + 2h\nu$

   with $h\nu = E_2 - E_1$
3. Life time of an atom $= 10^{-3}$ sec in a metastable state

   $= 10^{-8}$ sec in an excited state.
4. Population inversion : $N_2 > N_1$: Artificially created state.
5. Resonant cavity : Distance between the reflectors $= n\lambda / 2$, $n = 1, 2, 3$...
6. He - Ne Laser : Wavelength : 3.39 μm (infrared)

   6328 A° (visible)

   1.15 μm (infrared)
7. Nd - YAG Laser : Wavelength : 1.06 μm (infrared)
8. Semiconductor LASER : Wavelength : 8400 A° (infrared)
9. Holography : A two step technique of recording a 3D image using a beam.

   (i) Recording, (ii) Reconstruction.

EXERCISE

(A) Short Answer Type Questions

1. What is the full form of LASER?
2. State the important characteristics of LASER.
3. Why X-rays and LASER are so powerful than ordinary light?
4. Explain stimulated emission of radiation.
5. Define population inversion and its significance.
6. Differentiate between spontaneous emission and stimulated emission.
7. Define a metastable state and state its significance.
8. Diagrammatically explain the three level pumping scheme.
9. Diagrammatically explain the four level pumping scheme.
10. What is called the active medium?
11. What are the essential Components of a Laser Source?
12. What is the role of a resonant cavity in a Laser source?
13. Explain the role of a He in He - Ne Laser.
14. Explain the role of the ellipsoidal reflector in Nd -YAG laser.
15. How can a LED be converted to a Laser diode?
16. What is the fundamental principle of holography?
17. Why does not the hologram resemble the object?
18. Explain the necessary of a three level system for Laser production.

(B) Long Answer Type Questions

1. With simple diagrams explain the following terms :

   (i) Absorption, (ii) Spontaneous emission,

   (iii) Stimulated emission, (iv) Population inversion
2. What is called pumping and what is its significance in a LASER source. Explain diagrammatically the three level and four level pumping schemes. Describe different types of pumping methods.
3. With a neat energy level diagram explain the construction and working of a He - Ne laser. State its merits, demerits and applications.

4. With a neat energy level diagram explain the construction and working of Nd - YAG laser. State its application.
5. What is the principle of a semiconductor laser. Explain its working with energy diagram. State its applications.
6. What is holography and how does it differ from photography? Explain the process of recording and reconstruction of a hologram.
7. State various uses of LASER in defence, industrial and medical fields.

Previous University Examination Questions with Solutions

1. What does LASER stand for? In what respects it differ from an ordinary source of light? (M.U. May 2008, 14, 15, 18; Dec. 2019)
[Refer § 2.1]
2. Write full form of LASER. Explain main three processes involved in production of LASER with appropriate diagrams. (M.U. Dec 2006, 12; May 2009)
[Refer § 2.1, 2.2.1]
3. Differentiate between spontaneous emission and stimulated emission process related to Laser operation.
[Refer § 2.2.3] (M.U. May 2008, 12, 17; Nov. 2018; Dec 2010, 13, 14, 15, 19)
4. What is a population inversion state? Explain its significance in the operation of LASER. (M.U. May 2013, 17; Dec. 2016)
[Refer § 2.4.1]
5. What is pumping in LASER? Give the types of pumping.
[Refer § 2.4.2] (M.U. May 2009; Dec. 2016, 17)
6. Explain the terms : (M.U. Dec. 2008, 09, 10, 17)
(i) Spontaneous emission, (ii) Stimulated emission,
(iii) Metastable state, (iv) Population inversion, (v) Pumping.
[Refer § 2.2.1, 2.4.1, 2.4.2]
7. What is resonant cavity in the operation of a LASER? (M.U. May 2012)
[Refer § 2.4.4]
8. With neat energy level diagram describe the construction of a He - Ne Laser. What are its merits and demerits?
[Refer § 2.6.2] (M.U. May 2007, 08, 13, 15, 18; Nov. 2018; Dec. 2007, 14, 19)

9. Draw the energy level diagram of He - Ne Laser. What is its wavelength in visible range? (M.U. Dec. 2012) (3 m)
[Refer § 2.6.2]
10. Explain with neat diagram the construction and working of a Nd-YAG Laser.
[Refer § 2.6.3] (M.U. May 2011, 12; Dec. 08, 09, 19) (8 m)
11. Explain with neat diagrams the construction and working of a semiconductor diode laser. What serves the resonant cavity in semiconductor Laser?
[Refer § 2.6.4] (M.U. May 2010; Dec. 2012, 16) (7 m)
12. Write the difference LED and Laser diode.
[Refer § 2.6.4(e)] (M.U. May 2012; Nov. 2018) (7 m)
13. What is holography? Explain the construction and reconstruction of hologram. What is the difference between holography and photography?
[Refer § 2.7.1] (M.U. May 2009, 10, 13, 14, 17 ; Dec. 2011, 13, 15, 17, 19) (8 m)
14. Write a short note on holography. (M.U. May 2007; Dec. 2007, 10, 11, 16) (5 m)
[Refer § 2.7.1]
15. Differentiate between photography and holography. (M.U. May 2007, 19) (3 m)
[Refer § 2.7.2]

FIBRE OPTICS

2.8 Introduction

Fibre optics is a technology in which information is transmitted from one place to another with the help of an optical signal propagating through optical fibres. Optical fibres are used to transmit light signals over long distances.

An optical fibre is defined as a dielectric waveguide that confines light energy within its surface and guides it in a direction parallel to its axis.

2.9 Principle of Fibre Optics : Total Internal Reflection

The optical beam is made to travel through the optical fibre not by the simple mode of transmission but by the principle of total internal reflection.

Whenever a ray of light comes from a rarer medium (of refractive index μ_1) and enters a denser medium (of r.i., $\mu_2 > \mu_1$) it bends towards the normal as shown in

Fig. 2.19 (a). In this case, ... the angle of inci...

Snell's law is written as

$$\frac{\sin i}{\sin \theta} = \frac{\mu_2}{\mu_1} > 1$$ ......

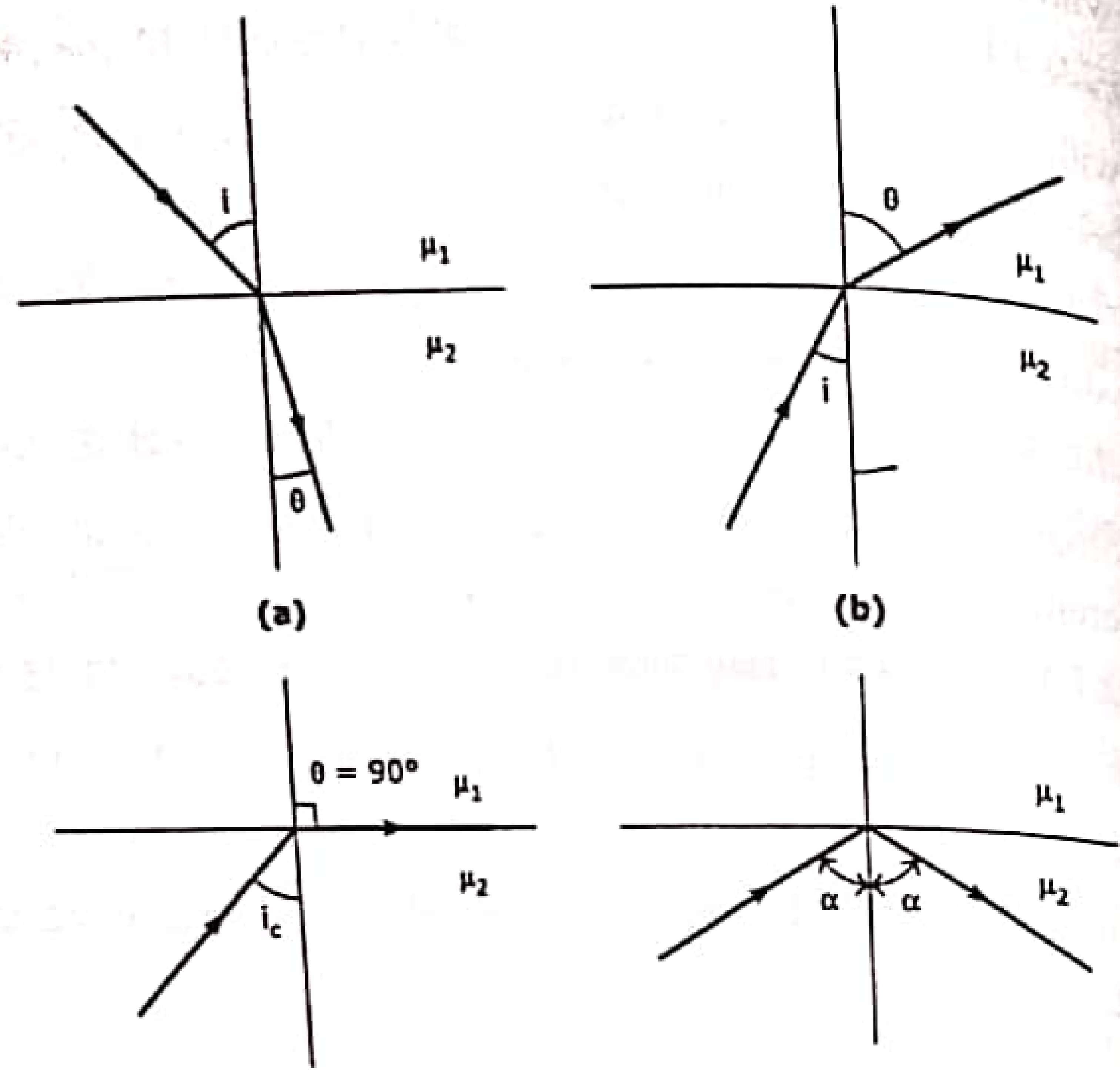


Fig. 2.19

On the otherhand, if a ray of light falls on a denser surface after passing thro... rarer medium, the refracted ray bends away from the normal on the inte... [Fig. 2.19 (b)] In this case

$$\theta > i$$

and Snell's law becomes

$$\frac{\sin i}{\sin \theta} = \frac{\mu_1}{\mu_2} < 1 \qquad \text{......(2.1...}$$

- Now, if the angle of incidence, i is gradually increased, the angle of refraction, θ increases and a time comes when θ becomes equal to 90° [See Fig. 2.19 (c)] ... angle of incidence for θ = 90° is called the ***critical angle, i_c***. In this case, Snell's ... is written as

$$\sin i_c = \frac{\mu_1}{\mu_2} < 1 \qquad \text{......(2...}$$

Finally, if a ray of light in denser medium is incident on the interface at an angle of incidence, $i > i_c$, the critical angle the light is reflected back into the denser medium [See Fig. 2.19 (d)]. This reflection is termed as ***total internal reflection***. The minimum angle of incidence for total internal reflection is

$$\alpha_{min} = i_c$$

and Snell's law becomes $$\sin \alpha_{min} = \frac{\mu_1}{\mu_2} \qquad \text{......(2.10-d)}$$

In ordinary reflection 4% of the incident energy is absorbed by the interface due to refraction and absorption at every incidence but in the case of total internal reflection total incident energy is reflected back to the medium.

This is why, using the principle of total internal reflection, optical signals are transmitted through optical fibres without any significant loss of energy. The emergent beam is as intense as the incident beam.

In a typical optical fibre about 2 m long, a ray undergoes about 45,000 reflections. Visible light can be transmitted successfully over a length of about 50 m through a single fibre.

For long distance transmission couplers are used to join several fibre pieces.

2.10 Basic Construction of Optical Fibres

The transmission properties of an optical fibre depends on its structural properties. In the most widely accepted structure, an optical fibre consists of an inner solid dielectric cylinder made up of high-silica-content glass known as the ***core*** of the fibre. The core is surrounded by a solid cylindrical dielectric shell, generally made up of law-silica-content glass or plastic. This is known as ***cladding*** as shown in Fig. 2.20.

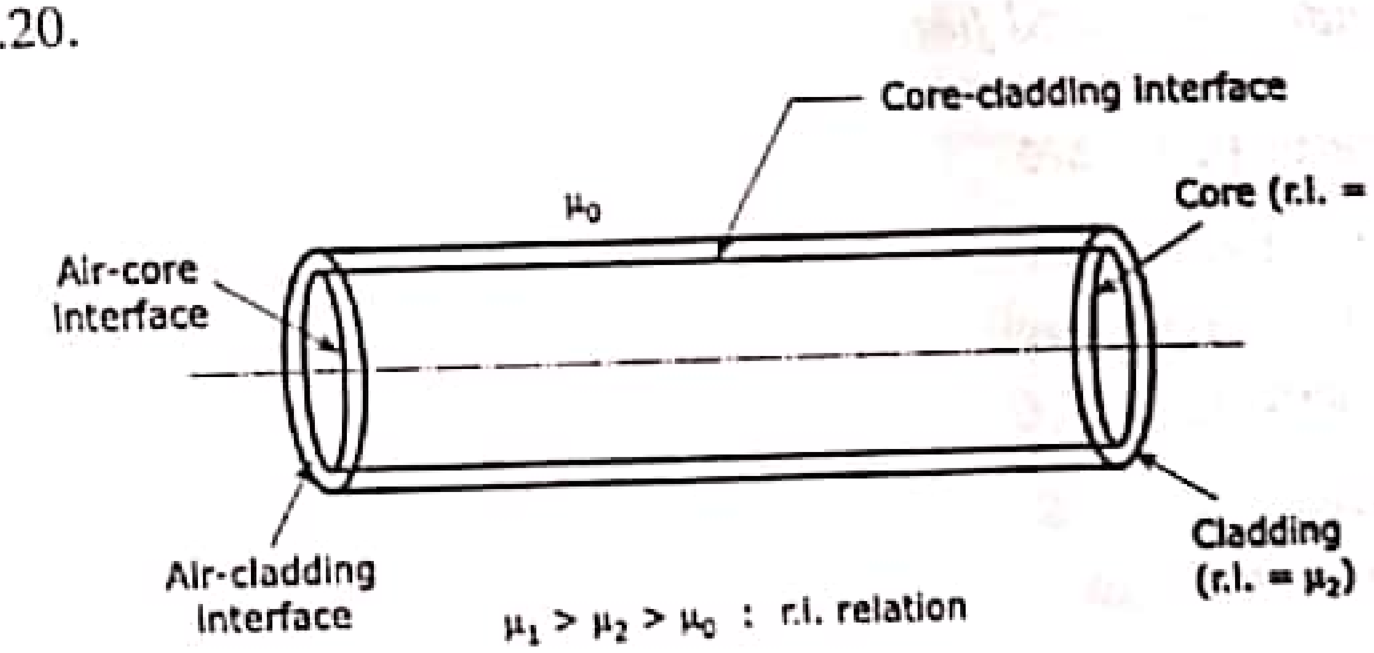
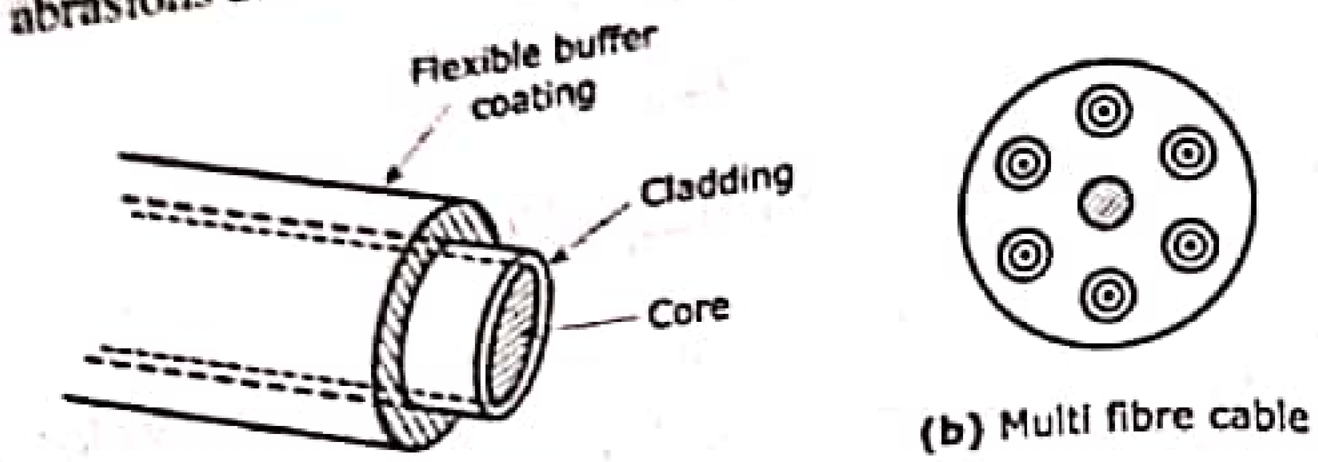


Fig. 2.20 : Structure of optical fibre

- The outermost region of an optical fibre is called the ***buffer coating***. It is a coating given to the cladding for extra protection. The buffer is elastic in nat... prevents abrasions as shown in Fig. 2.21.



(a) Single optical fibre (b) Multi fibre cable

Fig. 2.21

- Hence, the function of the three regions of an optical fibre can be summari... follows ;
  (i) Core : used to carry light.
  (ii) Cladding : confines the light to the core
  (iii) Buffer coating : protects the fibre from physical damage and environ... effects.

2.10.1 : Step Index and Graded Index Fibres and their Refractive Index Prof...

Optical fibres may be classified in terms of their refractive index prof... follows :

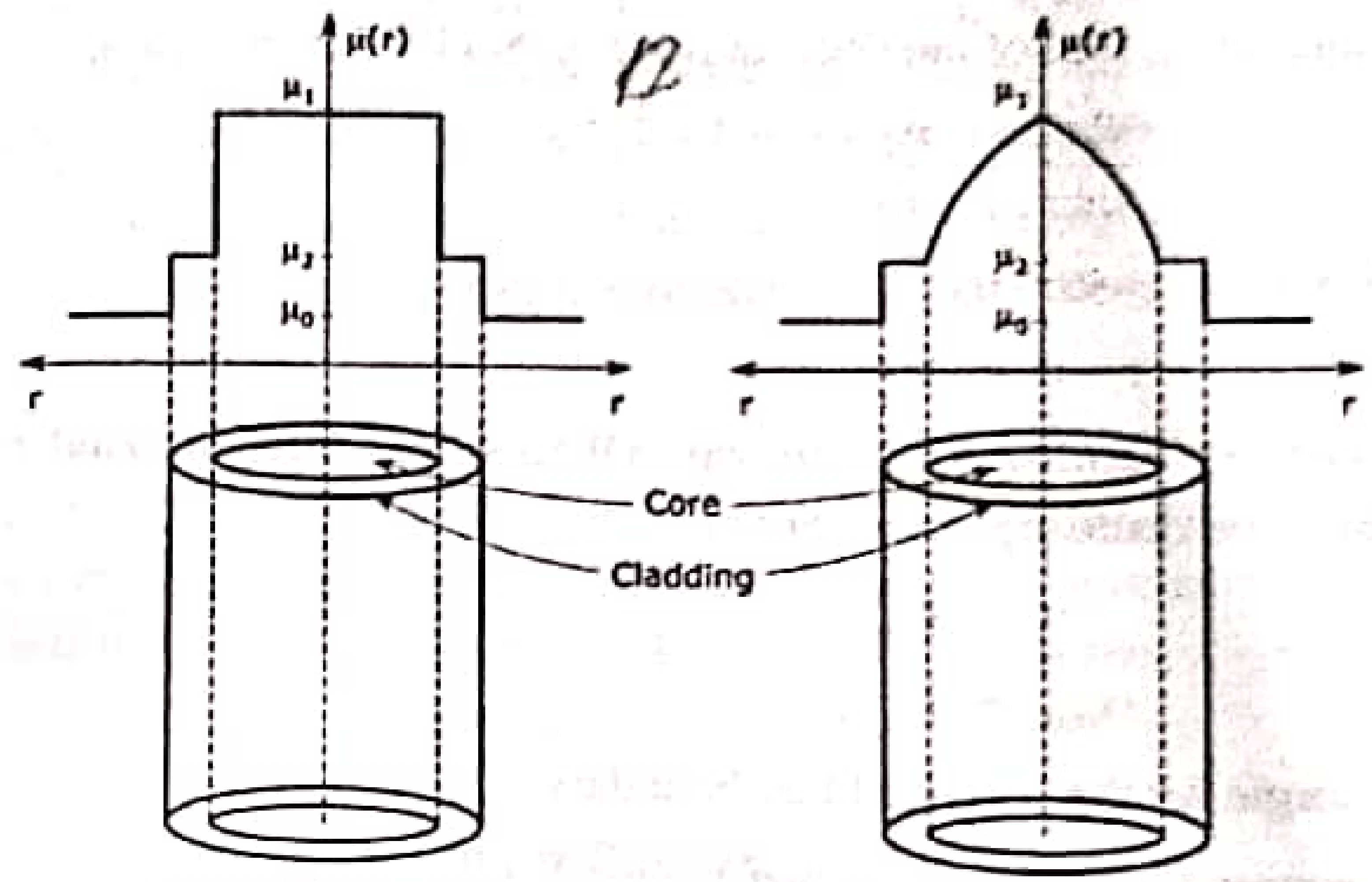
(A) Step Index (SI) Fibre

If the core refractive index remains constant at value μ_1 throughout the core and abruptly drops to the cladding refractive index μ_2 at the core-cladding bound... known as ***step index fibre*** or ***SI fibre*** as shown in Fig. 2.22 (a).

(B) Graded Index (GI) Fibre

If the core refractive index μ_1 varies as a function of the radial distance $\mu_1 = \mu_1(r)$ with the cladding refractive index constant at value μ_2, the fibre is c... ***graded index (GRIN) fibre*** or ***GI fibre*** as seen in Fig. 2.22 (b).

The diameter of a typical optical fibre ranges between 10 μm and 100 μm... overall diameter (core + cladding) ranges from 125 μm to 400 μm.



(a) Step index fibre (b) Graded index fibre

Fig. 2.22 : Refractive index profile

2.11 Numerical Aperture and Angle of Acceptance

2.11.1 : Propagation of Light through a Step-index Fibre

Consider a step index fibre of core refractive index, μ_1 and cladding refractive index, μ_2 such that $\mu_1 > \mu_2$. The refractive index of surroundings is μ_0, here.

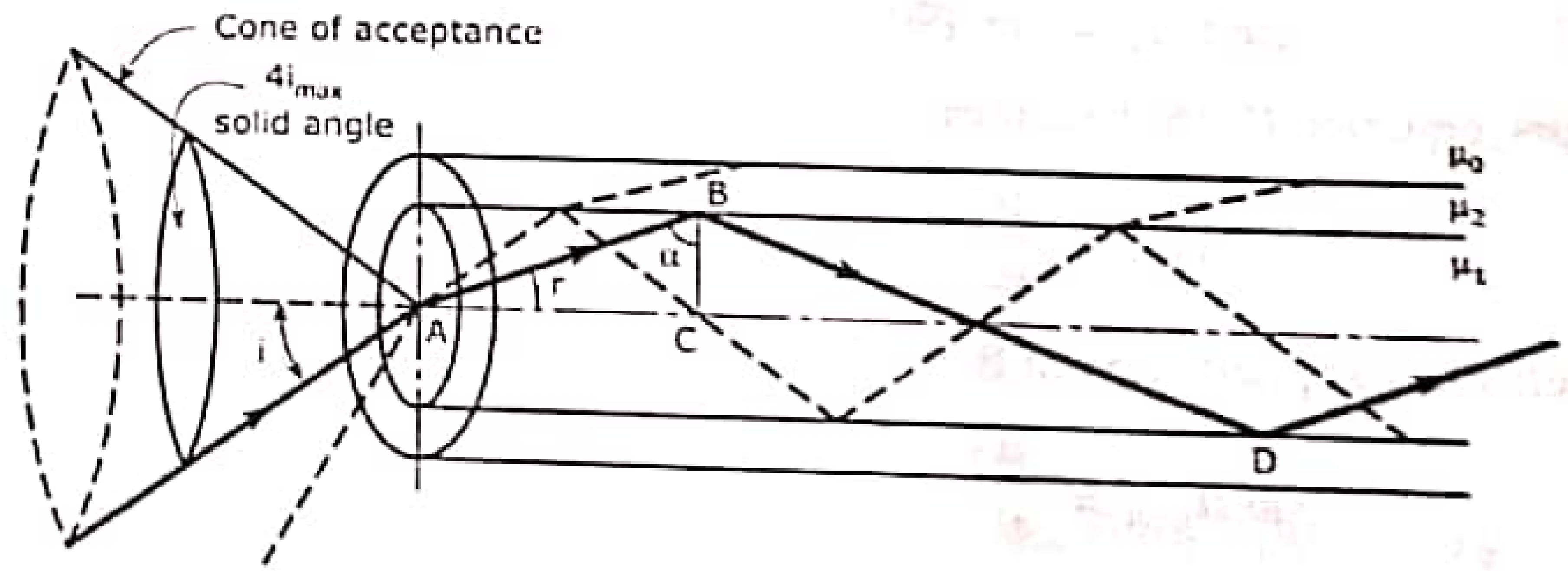


Fig. 2.23 : Propagation of light through S.I. fibre

Suppose a ray of light is incident at point A on the air-core boundary at an angle of incidence, i. The ray enters the core region at an angle of refraction, r and travels as AB. At point B on the core-cladding interface the ray is reflected along BD.

On the launching face of the fibre, at point A. Snell's law becomes,

$$\frac{\sin i}{\sin r} = \frac{\mu_1}{\mu_0} \quad \text{.........}$$

From Δ ABC, it is seen that

$$r + \alpha = 90° \quad \text{.........}$$

The necessary condition for the ray AB to suffer total internal reflection at point B on the core-cladding boundary is

$$\alpha > \alpha_{min} \quad \text{.........}$$

where $\alpha_{min} = i_c$ ....

is the critical angle for the core-cladding boundary.

From equation (2.12), it is seen that

$$r_{max} + \alpha_{min} = 90° \quad \text{.........}$$

Hence, for $\alpha = \alpha_{min}$, $r = r_{max}$ which requires $i = i_{max}$. Thus, equation (2.11) becomes

$$\frac{\sin i_{max}}{\sin r_{max}} = \frac{\mu_1}{\mu_0}$$

$$\sin i_{max} = \frac{\mu_1}{\mu_0} \sin r_{max} \quad \text{.........}$$

From equation (2.15), it is found that

$$\sin r_{max} = \sin(90° - \alpha_{min}) = \cos \alpha_{min}$$

Thus, equation (2.16) becomes

$$\sin i_{max} = \frac{\mu_1}{\mu_0} \cos \alpha_{min} \quad \text{.........}$$

Snell's law applied at point B on the core-cladding boundary reads

$$\sin \alpha_{min} = \frac{\mu_2}{\mu_1}$$

which gives $$\cos \alpha_{min} = \sqrt{1 - \frac{\mu_2^2}{\mu_1^2}} = \frac{\sqrt{\mu_1^2 - \mu_2^2}}{\mu_1} \quad \text{.........}$$

Combining equations (2.17) and (2.18), it can be written as

$$\text{N.A.} = \mu_0 \sin i_{max} = \sqrt{\mu_1^2 - \mu_2^2} \quad \text{.........}$$

defined as the ***numerical aperture*** (N.A.) of the optical fibre.

- All the rays launched at angles of incidence $i < i_{max}$ are totally internally reflected and transmitted through the fibre without any loss of intensity. The angle i_{max} is called the ***acceptance angle*** or ***angle of acceptance*** of a fibre.
- In three dimensions, the light rays contained within the cone of solid angle $4i_{max}$ are accepted and successfully transmitted along the fibre. Thus, the cone is called the *cone of acceptance*.
- Numerical aperture is a measure of the maximum angle of incidence, i_{max}, *i.e.*, the ***light gathering capacity*** of the fibre. As shown in Fig. 2.23, (with the dotted line), a ray incident at an angle greater than i_{max} strikes the core-cladding interface on an angle less than the critical angle, i_c. Hence, at every incidence it is partially refracted through the cladding and gradually loses all its optical energy.

2.11.2 : Fractional Refracture Index Change

- The fractional refracture index change is defined as the fractional difference between the refractive indices of the core and the cladding. It is expressed as

$$\Delta = \frac{\mu_1 - \mu_2}{\mu_1} \quad \text{...................} (2.20)$$

- Since always, $\mu_1 > \mu_2$ for the requirement of total internal reflection, $\Delta << 1$. **Typically, Δ is of the order of 0.01.**
- Equation (2.19) and (2.20) can be combined to express the numerical aperture of the step-index fibre as

$$\text{NA} = \sqrt{\mu_1^2 - \mu_2^2} = \sqrt{(\mu_1 + \mu_2)(\mu_1 - \mu_2)}$$

Here, since μ_1 is little larger than μ_2 $(\mu_1 - \mu_2)$ is small and $(\mu_1 + \mu_2) \equiv 2\mu_1$.

Hence, $$\text{NA} = \sqrt{2\mu_1(\mu_1 - \mu_2)} = \sqrt{2\mu_1 \Delta \mu_1}$$

$$\text{NA} = \mu_1 \sqrt{2\Delta} \quad \text{...................} (2.21)$$

2.11.3 : Propagation of Light through a Graded Index Fibre (GRIN Fibre)

In the GRIN fibre the core refractive index decreases continuously with radial distance from the fibre axis but is generally constant in the cladding.

Consider a GRIN fibre in which the core is imagined to be divided into individual layers of refractive indices $\mu_1 > \mu_2 > \mu_3 >$ ..... These layers are so then that the refractive index variation seems to be continuous.

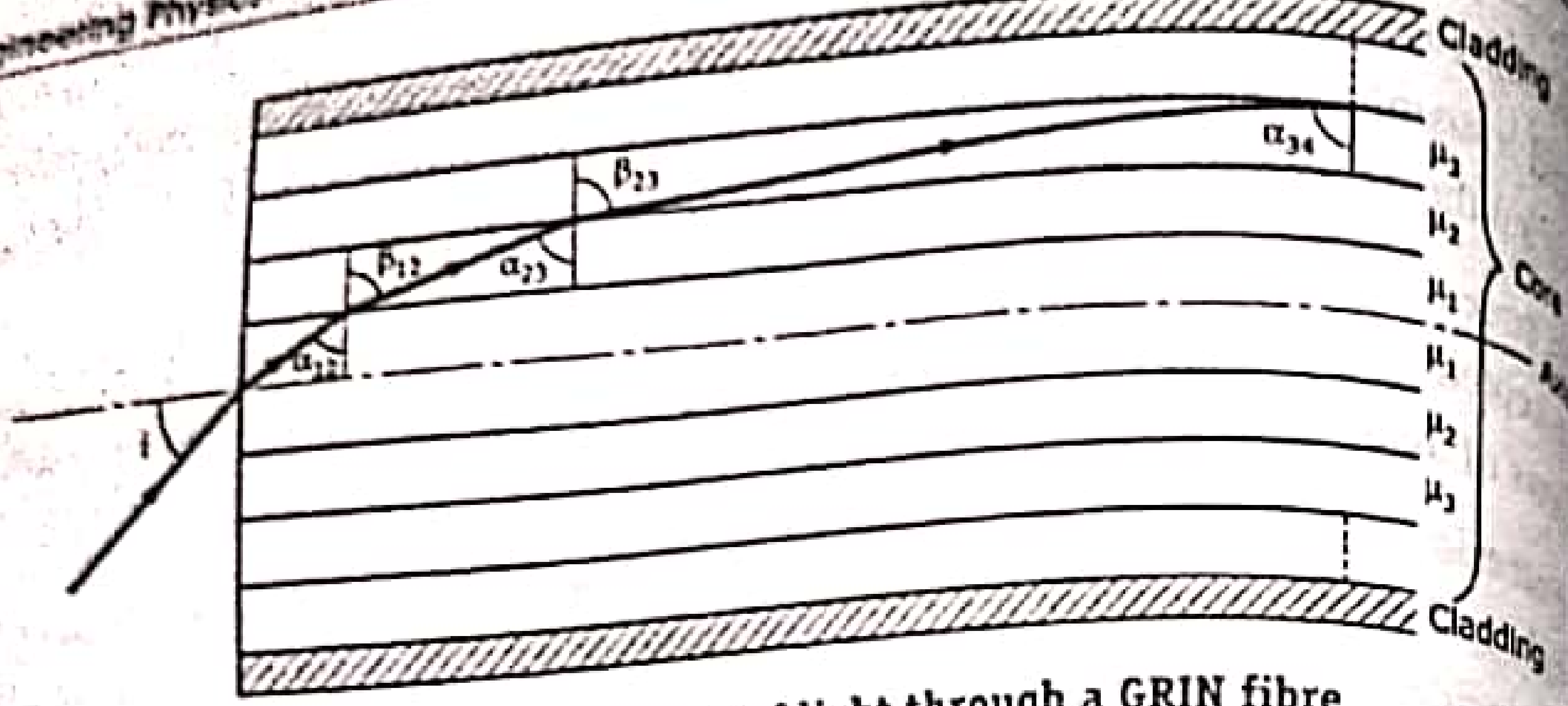


Fig. 2.24 : Transmission of light through a GRIN fibre

Let a ray of light enter the innermost layer of the core region of refractive ind and strike the layer 1-layer 2 interface at an angle α_{12}. As $\mu_2 < \mu_1$, the ray is ref through an angle $\beta_{12} > \alpha_{12}$. Then it strikes the layer 2-layer 3 interface at an angle is refracted through an angle $\beta_{23} > a_{23}$ since $\mu_3 < \mu_2$.

Applying Snell's law at point A, it is found that

$$\frac{\sin \alpha_{12}}{\sin \beta_{12}} = \frac{\mu_2}{\mu_1}$$

$$\therefore \quad \mu_1 \sin \alpha_{12} = \mu_2 \sin \beta_{12} \quad \text{.............(}$$

Similarly, at point B Snell's law gives

$$\frac{\sin \alpha_{23}}{\sin \beta_{12}} = \frac{\mu_2}{\mu_2}$$

$$\mu_2 \sin \alpha_{23} = \mu_3 \sin \beta_{23} \quad \text{.............(}$$

From Fig. 2.24, it is found that

$$\beta_{12} = \alpha_{23} \text{ (alternate)} \quad \text{.............(2}$$

and $$\beta_{23} = \alpha_{34} \text{ (alternate)} \quad \text{.............(2}$$

Hence, equations (2.22), (2.23) and (2.24) being combined becomes

$$\mu_1 \sin \alpha_{12} = \mu_2 \sin \alpha_{23} = \mu_3 \sin \alpha_{34}$$

$$\mu(r) \sin \alpha(r) = \text{constant} \quad \text{.............(}$$

This shows that in the core region as μ decreases with r, α (r) increases u reaches the critical angle i_c for any interface. Hence, the ray gradually bends toward axis until it suffers total internal reflection at some interface of the core region. Th never reaches the core-cladding boundary.

The refractive index function of a GRIN fibre is given by

$$\mu(r) = \mu_1 \sqrt{1 - 2\Delta (r^2/a^2)^2}, \qquad r < a, \text{ inside core}$$
$$= \mu_2, \qquad r > a, \text{ in cladding}$$

where 'a' is the radius of the core.

The numerical aperture of a GRIN fibre is written as

$$\text{N.A.} = \sqrt{\mu^2(r) - \mu_2^2}$$
$$= \sqrt{\mu_1^2 \left\{1 - 2\Delta (r^2/a^2)^2\right\} - \mu_2^2}$$
$$= \sqrt{(\mu_1^2 - \mu_2^2) - 2\mu_1^2 \Delta (r^2/a^2)^2}$$
$$\therefore \quad \text{N.A.} = \sqrt{(\mu_1 + \mu_2)(\mu_1 - \mu_2) - 2\mu_1^2 \Delta (r^2/a^2)^2}$$

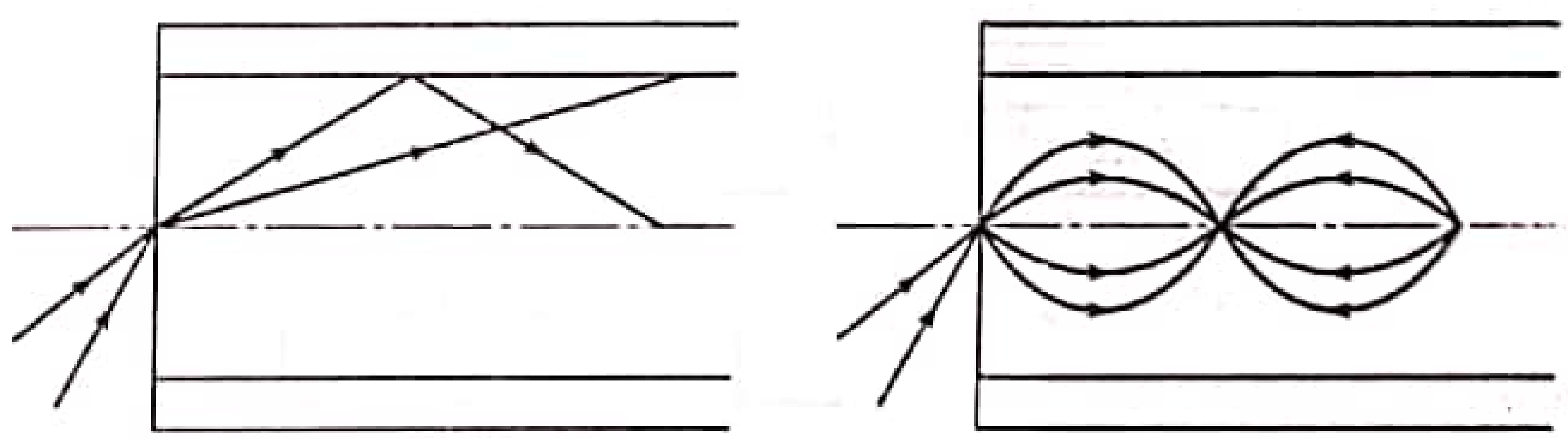
Assuming $\mu_1 + \mu_2 \simeq 2\mu_1$, it canbe written that

$$\text{N.A.} = \sqrt{2\mu_1 (\mu_1 - \mu_2) - 2\mu_1^2 \Delta (r^2/a^2)^2}$$
$$= \sqrt{2\mu_1^2 \Delta - 2\mu_1^2 \Delta (r^2/a^2)^2} \quad \text{.... [Using Eqn. (2.20)]}$$
$$\text{N.A.} = \mu_1 \sqrt{2\Delta} \sqrt{1 - (r^2/a^2)^2} \quad \text{.................... (2.26)}$$

2.11.4 : Comparison of Propagation of Light through Step-index and Graded Index Fibres

The paths travelled by a ray in a SI and in a GI fibres are completely different.

In a step index fibre the ray changes its direction of motion abruptly at each total internal reflection whereas in a GI fibre the direction changes gradually by successive refraction between the total internal reflections.



(a) S.I. Fibre (b) G.I. Fibre

Fig. 2.25 : Propagation of light

- In a SI fibre as shown in Fig. 2.25 (a) different rays launched at different follow different paths and emerges at different times from different points output end. This causes pulse dispersion.
- On the otherhand, as seen in Fig. 2.25 (b) rays entering the GI fibre at dif angles follow different paths with the same time period. Thus, there is a p self-focussing of the rays and pulse dispersion is very small.

2.12 Mode of Propagation

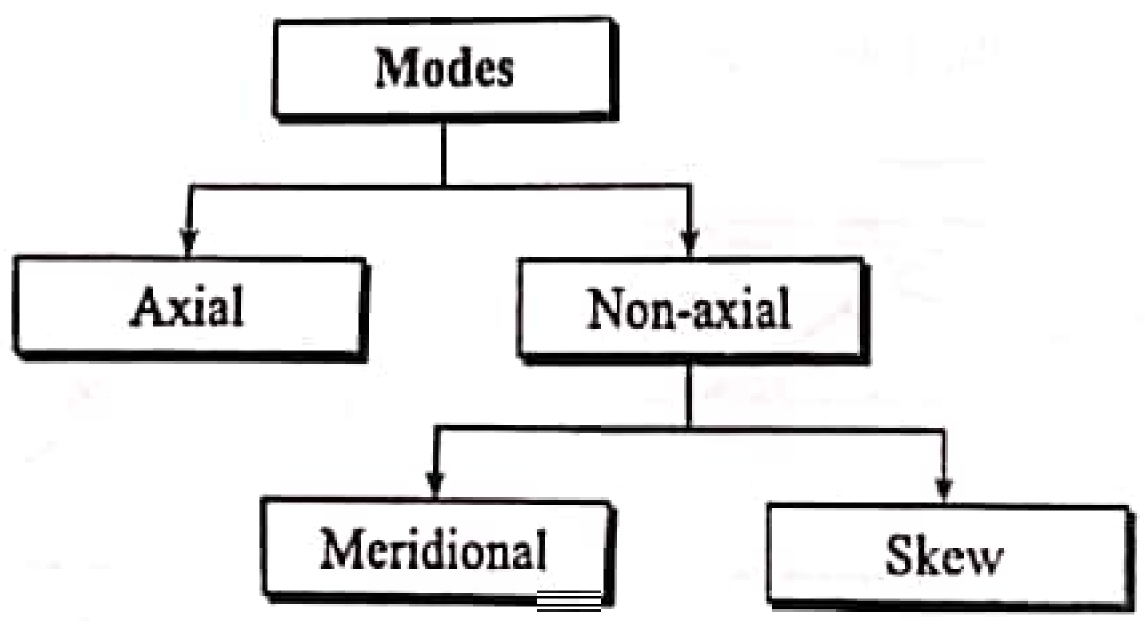
- In simple context, *a mode of propagation in an optical fibre is an allowed followed by more than one ray.*
- Even in a perfectly constructed optical fibre it is seen that all the rays la through the cone of acceptance are not guided to the output end.
- This happens because while travelling through the optical fibre the rays in with each other. Some groups of rays interfere constructively and are inte while other groups of rays interfere destructively and fade out.
- Every group of rays that are intensified follow a single path called a *mode*.

As a result, the fibre allows only a few selected paths *i.e.*, a finite number of m

The number of allowed modes depends on the ratio d / λ where 'd' is the diame the core and λ is the wavelength of the guided light.

2.12.1 : Classification of Modes

The modes can be classified as follows :



Different modes are explained as follows :

- **Axial modes :** These are the modes propagating along or parallel to the

- **Non-axial modes :** These are the modes that undergo successive total internal reflections.
- **Meridional modes :** These modes follow zigzag path and repeatedly cross the axis at regular intervals.
- **Skew modes :** These modes are helical about the axis but never cross the axis.

The modes are illustrated in Fig. 2.26.

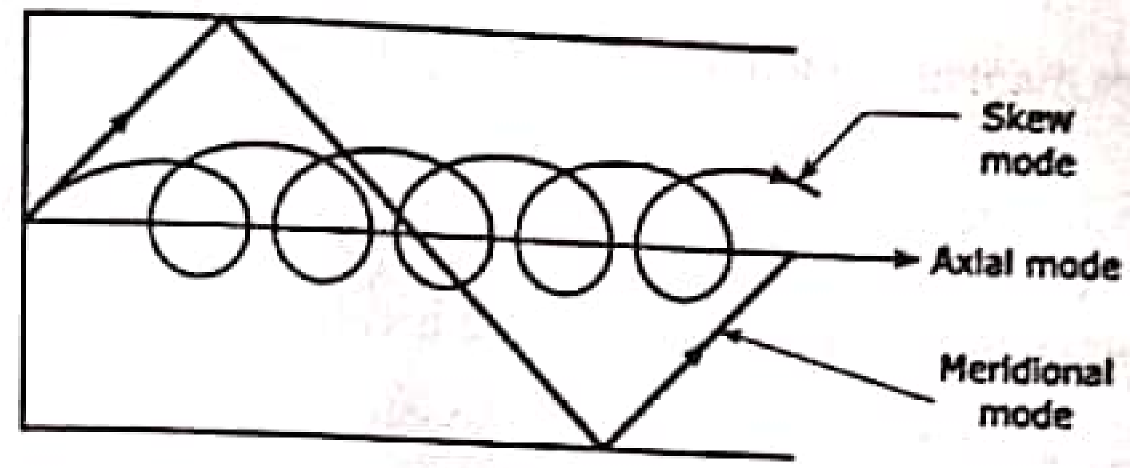


Fig. 2.26 : Types of modes

2.12.2 : Maximum Number of Allowed Modes (N) and V Number

In an optical fibre of uniform structure

- The axial modes are called zero order modes.
- The modes that propagate with $\alpha = i_c$ (Fig. 2.27) are the highest order modes.
- The modes with $\alpha > i_c$ are the medium order modes.

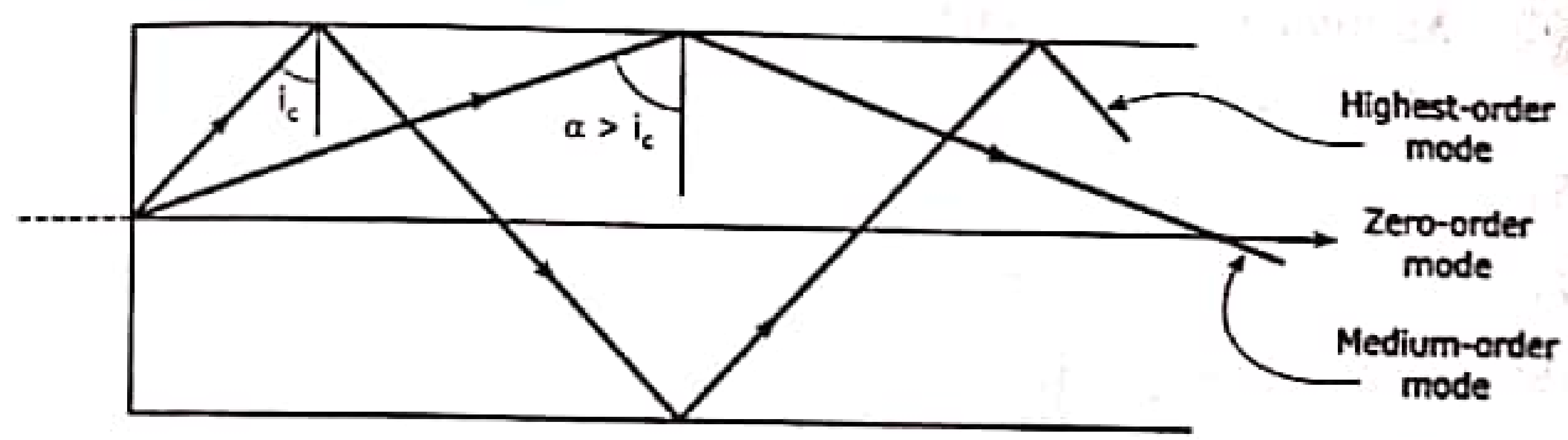


Fig. 2.27 : Order of modes

- The zero order mode is the fastest mode and the highest order mode is the slowest mode. Hence, they reach the output end at different times.
- The number of modes that an optical fibre can support is determined by its V number also called its *normalized frequency*. It is given by

$$V = \frac{\pi d}{\lambda} N.A. = \frac{2\pi a}{\lambda}\sqrt{\mu_1^2 - \mu_2^2} \qquad \text{.................... (2.27)}$$

where 'λ' is the wavelength of the guided light,

'a' is the core radius, and

'd' is the core diameter.

The maximum number of modes supported by a step index fibre is

$$N_m = \frac{V^2}{2} \quad \text{.......... (}$$

The maximum number of modes supported by a graded index fibre is

$$N_m = \frac{V^2}{4} \quad \text{.......... (}$$

The wavelength corresponding to V = 2.405 is called the ***cut-off wavelength.***

Every fibre has a specific V number and a specific cut-off wavelength.

For V < 2.405, the fibre can support only one mode *i.e.*, a single mode fibre.

For V > 2.405, the fibre can support many modes *i.e.*, a multimode fibre.

2.12.3 : Classification of Optical Fibres

According to the number of propagating modes optical fibres are classified as :

(i) ***Single Mode Fibre (SMF)*** : A single mode fibre has a small core diam *and can support only one mode of propagation i.e.*, the zero order mode

(ii) **Multimode Fibre (MMF)** : Multimode fibres have larger core diamete compared to SMF and can allow a number of modes.

The Overall Classification of Optical Fibres and as follows :

(a) Single mode step-index fibre (SMSIF)

(b) Multimode step-index fibre (MMSIF)

(c) Multimode graded index fibre (MMGIF)

The single mode graded index fibre (SMGIF) is not possible to manufacture bec the graduation of the refractive index is not possible in a narrow core.

The structural and transmission characteristics of these three types of optical f is shown in Fig. 2.28.

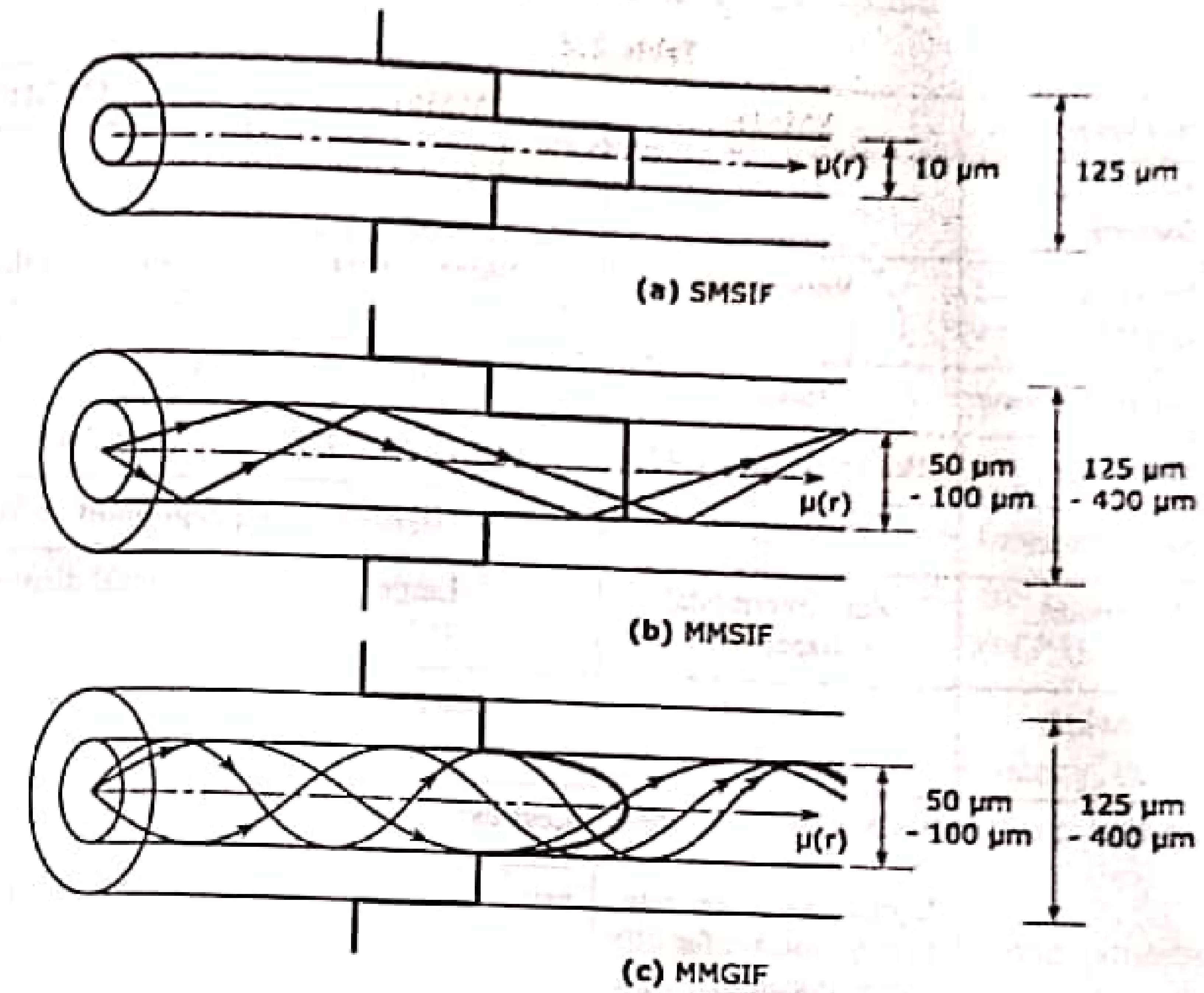


Fig. 2.28 : SMF and MMF

2.12.4 : Comparison of Different Types of Optical Fibres

The single mode step index fibre (SMSIF), the multimode step index fibre (MMSIF) and the multimode graded index fibre (MMGIF) are compared as shown in Table 2.2.

2.13 Applications of Optical Fibres

Optical fibres have wide range of applications. They have many advantages over their metallic equivalents due to their various merits listed below.

1. These are cheaper than metallic conductors.
2. These are smaller in size, lighter in weight and more flexible yet strong as compared to conducting materials.
3. Optical fibres are made up of dielectric materials. So these are not affected by high voltage lines.
4. In optical fibres, information is carried by photons through dielectric media. Hence it is not affected by external electric of magnetic fields. Also the information is transmitted in a very secured way.

Table 2.2

Sr. No.	Characteristics	SMSIF	MMSIF	MMGIF
1.	Typical core diameter	10 μm	50 to 100 μm	50 to 100 μm
2.	Numerical aperture	Very small	Large (< 0.5)	Smaller than that of MMSIF
3.	No. of modes	Only one	Many	Many
4.	V-number	Between 0 and 2.475	Greater than 2.475	Greater than 2.475
5.	Attenuation	Least	High	Optimum
6.	Dispersion	Zero intermodal dispersion	Large	Material dispersion dominant
7.	Bandwidth	> 3 GHz-km	< 200 MHz-m	200 MHz-km - 300 Hz - km
8.	Merits	No degradation of the information signal high data transfer rate, highly suitable for communication	Less expensive Launching of light is easier using LED or Laser sources. Coupling of fibres is easier.	Launching of light easy by using LED, Laser sources.
9.	Demerits	Expensive; Launching of light is difficult. A Laser source is necessary. Coupling of fibre is difficult.	Signal degradation occurs. Less suitable for communications.	Most expensive due to its complex structure. Coupling of fibre to the light source is difficult.
10.	Applications	Under-water cables	Data links	Telephone lines

5. Information carrying capacity of an optical fibre is very large as compared metallic cable.

6. Due to the transmission by total internal reflection the loss of energy thr optical fibres is very small.

2.13.1 : Fibre Optic Communication System

- Communication may be defined as the transfer of information from one pla another. For this a communication system is necessary.
- Within a communication system the information signal is superimposed on a ca wave and the carrier wave is modulated by the information signal. The modu carrier wave is then transmitted through the communication channel to the destin where it is received and demodulated to extract the original information.

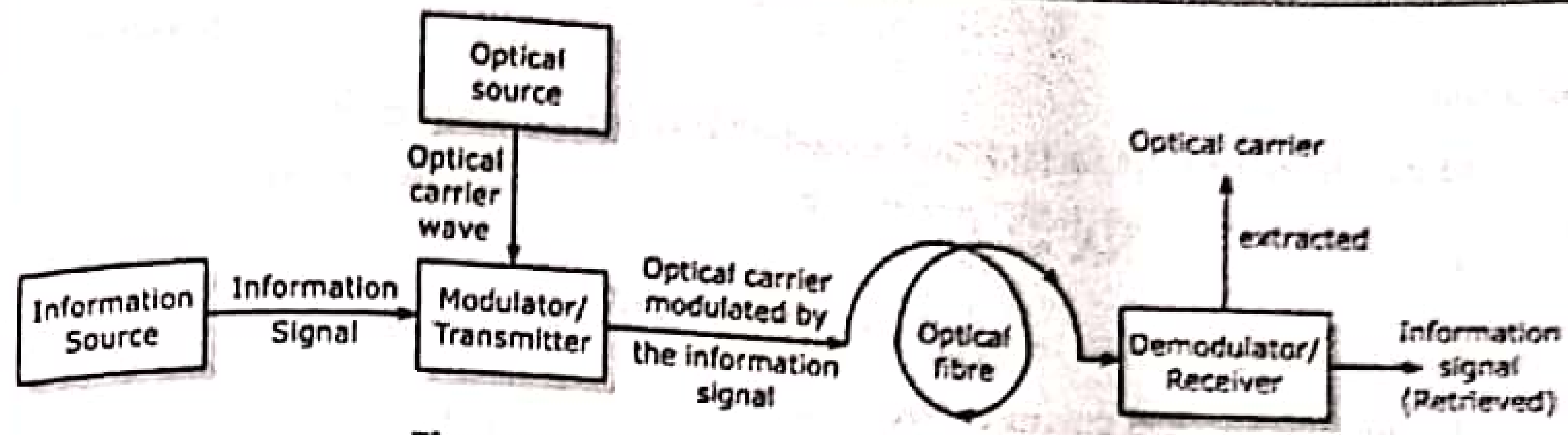


Fig. 2.29 : Optical fibre communication system

- The carrier waves are electromagnetic waves. Earlier there has been a frequent use of either the radio waves (frequency ~ 3 kHz to 300 GHz), the microwaves (frequency ~ 3 GHz to 30 GHz) or the millimeter waves (frequency ~ 30 GHz to 300 GHz), as a carrier wave.
- It has been found theoretically that the greater the carrier frequency, the larger is the transmission bandwidth and thus the information carrying capacity of the communication system.
- After the advent of laser in 1960, communication has become possible with an electromagnetic carrier selected from the optical range of frequencies.
- At higher optical frequencies (~ 10^{15} Hz) a large frequency bandwidth (~ 10^4 times the bandwidth available with a microwave carrier signal) and a high information carrying capacity (~ 10^5 times the information carrying capacity of a microwave carrier signal) are available.
- However, light energy gets dissipated in open atmosphere by inverse square law,

$$I \propto \frac{1}{d^2}$$

where 'I' is the intensity of the light beam and 'd' is the distance travelled.
This dissipation is caused by the diffraction and scattering of light by dust particles, water vapour etc. and due to absorption in the medium.
- Hence, to transmit an optical carrier signal over a long distance a guiding channel is required. This is done by sending an optical beam or pulse through an optical fibre.

2.14 Solved Problems

Problem 1

Calculate the numerical aperture of an optical fibre with core refractive index 1.55 and cladding refractive index 1.53.

Solution :

Data : $\mu_1 = 1.55,\ \mu_2 = 1.53$

Formula : $NA = \sqrt{\mu_1^2 - \mu_2^2}$

Calculations : $NA = \sqrt{(1.55)^2 - (1.53)^2} = 0.248$

Result : Numerical aperture, NA = 0.248.

Problem 2

Calculate the numerical aperture and hence the acceptance angle for an optical f[illegible] Given that the refractive indices of the core and the cladding are 1.45 and 1.40 respecti[illegible]

(M.U. May 2009; Dec. 2012)

Solution :

Data : $\mu_1 = 1.45,\ \mu_2 = 1.40$

Formula : $NA = \sin i_{max} = \sqrt{\mu_1^2 - \mu_2^2}$

Calculations :

$$NA = \sqrt{(1.45)^2 - (1.40)^2} = 0.3775$$

$$i_{max} = \sin^{-1}(NA) = \sin^{-1}(0.1425)$$

$$\therefore \quad i_{max} = 22.17°$$

Result : Numerical aperture, NA = 0.3775

Acceptance angle, $i_{max} = 22.17°$

Problem 3

A fibre cable has an acceptance angle of 30° and core refractive index of 1.4. Cal[illegible] the refractive index of cladding. **(M.U. Nov. 2016; May 2018)**

Solution :,

Data : $i_{max} = 30°,\ \mu_1 = 1.4,\ \mu_2 = ?$

Formula : $\sin i_{max} = \sqrt{\mu_1^2 - \mu_2^2}$

Calculations :

$$\mu_2 = \mu_1^2 - (\sin i_{max})^2$$

$$= (1.4)^2 - (\sin 30°)^2$$

$$\therefore \quad \mu_2 = (1.4)^2 - (0.5)^2 = 1.71$$

Result : Refractive index of cladding = 1.71.

Problem 4

The N.A. of an optical fibre is 0.5 and the core refractive index is 1.54. Find the refractive index of cladding. **(M.U. Nov. 2014; May 2017) (3 m)**

Solution :

Data : N.A. = 0.5, $\mu_1 = 1.54$

Formula : $N.A. = \sqrt{\mu_1^2 - \mu_2^2}$

Calculations :

$$\mu_2 = \sqrt{\mu_1^2 - (N.A.)^2}$$

$$= \sqrt{(1.54)^2 - (0.5)^2}$$

$$\therefore \quad \mu_2 = 1.456$$

Result : Refractive index of cladding = 1.456.

Problem 5

An optical fibre has a NA of 0.20 and the refractive index of cladding is 1.59. Determine the core refractive index and the acceptance angle for the fibre in water which has a refractive index of 1.33. **(M.U. May 2010, 11, 13, 17) (5 m)**

Solution :

Data : NA = 0.20, $\mu_2 = 1.59,\ \mu_0 = 1.33$

Formula : $NA = \mu_0 \sin i_{max} = \sqrt{\mu_1^2 - \mu_2^2}$

Calculations :

$$NA = \sqrt{\mu_1^2 - \mu_2^2}$$

$$\mu_1 = \sqrt{(NA)^2 + \mu_2^2}$$

$$= \sqrt{(0.2)^2 + (1.59)^2}$$

$$\therefore \quad \mu_1 = 1.6025$$

$$i_{max} = \sin^{-1}\frac{NA}{\mu_0} = \sin^{-1}\frac{0.2}{1.33}$$

$$\therefore \quad i_{max} = 8.64°$$

Result : Core refractive index, $\mu_1 = 1.6025$

Acceptance angle, $i_{max} = 8.64°$.

Problem 6

A typical relative refractive index difference for an optical fibre is 1 %. Estimate th numerical aperture and the critical angle at the core-cladding interface if the core refractiv index is 1.46.

Solution :

Data : $\Delta = 0.01,\ \mu_1 = 1.46$

Formulae :

$$\Delta = \frac{\mu_1 - \mu_2}{\mu_1}$$

$$NA = \mu_1\sqrt{2\Delta}$$

$$\sin\alpha_{min} = \frac{\mu_2}{\mu_1}$$

Calculations :

$$NA = 1.46\sqrt{2\times 0.01} = 0.2064$$

$$\Delta = \frac{\mu_1 - \mu_2}{\mu_1} = 1 - \frac{\mu_2}{\mu_1}$$

$$\frac{\mu_2}{\mu_1} = 1 - \Delta = 1 - 0.01 = 0.99$$

$$\alpha_{min} = \sin^{-1}\frac{\mu_2}{\mu_1} = \sin^{-1} 0.99 = 81.89°$$

Result : Numerical aperture, $NA = 0.2064$

Critical angle, $\alpha_{min} = 81.89°$.

Problem 7

A glass material A with which an optical fibre is made has a refractive index of 1.5 This material is clad with another material B whose refractive index is 1.51. The light the fibre is launched from air. Calculate the numerical aperture of the fibre.

(M.U. May 2013; Nov. 2018) (3

Solution :

Data : $\mu_0 = 1.0,\ \mu_1 = 1.55,\ \mu_2 = 1.51$

Formula :

$$NA = \sqrt{\mu_1^2 - \mu_2^2}$$

Calculations :

$$NA = \sqrt{(1.55)^2 - (1.51)^2} = 0.349$$

Result : Numerical aperture, $NA = 0.349$.

Problem 8

Calculate the refractive index of the core and the cladding of an optical fibre with numerical aperture 0.22 and fractional refractive index change 0.012.

(M.U. Dec. 2017) (3 m)

Solution :

Data : $NA = 0.22,\ \Delta = 0.012.$

Formula :

$$NA = \sqrt{\mu_1^2 - \mu_2^2} = \mu_1\sqrt{2\Delta}$$

Calculations :

$$NA = \mu_1\sqrt{2\Delta}$$

$$\mu_1 = \frac{NA}{\sqrt{2\Delta}} = \frac{0.22}{\sqrt{2\times 0.012}} = 1.42$$

$$NA = \sqrt{\mu_1^2 - \mu_2^2}$$

$$\mu_2 = \sqrt{\mu_1^2 - (NA)^2} = \sqrt{(1.42)^2 - (0.22)^2} = 1.419$$

Result : Core refractive index, $\mu_1 = 1.42$,

Cladding refractive index, $\mu_2 = 1.419$.

Problem 9

The refractive index of core and cladding of a SI fibre are 1.52 and 1.41 respectively. Calculate (i) critical angle, (ii) N.A., and (iii) the maximum incidence angle.

(M.U. Dec. 2016) (7 m)

Solution :

Data : $\mu_1 = 1.52,\ \mu_2 = 1.41$

Formulae :

$$\sin\alpha_{min} = \frac{\mu_2}{\mu_1}, \qquad NA = \sqrt{\mu_1^2 - \mu_2^2}$$

$$\sin i_{max} = N.A.$$

Calculations :

$$\alpha_{min} = \sin^{-1}\left(\frac{1.41}{1.52}\right) = 68.06°$$

$$N.A. = \sqrt{(1.52)^2 - (1.41)^2} = 0.5677$$

$$i_{max} = \sin^{-1}(0.5677) = 34.59°$$

Results : Critical angle, $\alpha_{min} = 65.06°$,

Numerical aperture, N.A. = 0.5677,

Maximum incidence angle, $i_{max} = 34.59°$.

Problem 10

A light ray enters an optical fibre from air (r.i. = 1.0). The fibre has core refractive index 1.5 and cladding refractive index 1.48. Find the

(i) Critical angle, (ii) Fractional refractive index,

(iii) Acceptance angle, and (iv) Numerical aperture.

Solution :

Data : $\mu_0 = 1,\ \mu_1 = 1.5,\ \mu_2 = 1.48$

Formulae :

$$\sin i_c = \frac{\mu_2}{\mu_1}, \qquad \Delta = \frac{\mu_1 - \mu_2}{\mu_1}$$

$$i_{max} = \sin^{-1}\frac{\sqrt{\mu_1^2 - \mu_2^2}}{\mu_0}, \qquad NA = \sqrt{\mu_1^2 - \mu_2^2}$$

Calculations :

(i) $$i_c = \sin^{-1}\left(\frac{\mu_2}{\mu_1}\right) = \sin^{-1}\left(\frac{1.48}{1.5}\right) = 80.63°$$

(ii) $$\Delta = \frac{\mu_1 - \mu_2}{\mu_1} = \frac{1.5 - 1.48}{1.5} = 0.0133$$

(iii) $$i_{max} = \sin^{-1}\frac{\sqrt{\mu_1^2 - \mu_2^2}}{\mu_0} = \sin^{-1}\frac{\sqrt{(1.5)^2 - (1.48)^2}}{1} = 14.13°$$

(iv) $$NA = \sqrt{\mu_1^2 - \mu_2^2} = \sqrt{(1.5)^2 - (1.48)^2} = 0.244$$

Result : Critical angle, $i_c = 80.63°$,

Fractional refractive index, $\Delta = 0.0133$,

Acceptance angle, $i_{max} = 14.13°$,

Numerical aperture, $NA = 0.244$.

Problem 11

A glass clad fibre is made with core glass of r.i. 1.5 and the cladding is doped to give a fractional index difference of 0.0005. Find

(i) The refractive index of the cladding,

(ii) The critical internal reflection angle,

(iii) The critical external acceptance angle

(iv) The numerical aperture

(M.U. May 2003) (5 m)

Solution :

Data : $\mu_1 = 1.5,\ \Delta = 0.0005$

Formula :

$$\Delta = \frac{\mu_1 - \mu_2}{\mu_1},$$

$$\sin i_{max} = \sqrt{\mu_1^2 - \mu_2^2}$$

$$NA = \sqrt{\mu_1^2 - \mu_2^2}$$

Calculations :

(i) $$\mu_2 = \mu_1(1 - \Delta) = 1.5(1 - 0.0005) = 1.49925$$

(ii) $$\alpha = \sin^{-1}\frac{\mu_2}{\mu_1} = \sin^{-1}\frac{1.49925}{1.5} = 88.18°$$

(iii) $$i_{max} = \sin^{-1}\sqrt{\mu_1^2 - \mu_2^2}$$

$$i_{max} = \sin^{-1}\sqrt{(1.5)^2 - (1.49925)^2} = 2.718°$$

(iv) $$NA = \sqrt{(1.5)^2 - (1.49925)^2} = 0.474$$

Result : The cladding r.i., $\mu_2 = 1.49925$

The critical internal reflection angle, $\alpha = 88.18°$

The critical external acceptance angle, $i_{max} = 2.718°$

The numerical aperture , $NA = 0.474$

Problem 12

An optical glass fibre of refractive index 1.50 is to be clad with another glass to ensure internal reflection that will contain light travelling within 5° of the fibre axis. What maximum index of refraction is allowed for the cladding? **(M.U. May 2014) (3 m)**

Solution :

Data : $\mu_1 = 1.5,\ i_{max} = 5°,\ \mu_o = 1$

Formula : $$\mu_o \sin i_{max} = \sqrt{\mu_1^2 - \mu_2^2}$$

Calculations :

$$\sin^2 i_{max} = \mu_1^2 - \mu_2^2$$

$$\mu_2 = \sqrt{\mu_1^2 - \sin^2 i_{max}}$$

$$= \sqrt{(1.5)^2 - (\sin 5^\circ)^2}$$

$$= \sqrt{(1.5)^2 - (0.087)^2}$$

$$\therefore \quad \mu_2 = 1.498$$

Result : Maximum allowed cladding refractive index = 1.498.

Problem 13

Consider a multimode step index fibre with $\mu_1 = 1.53$, $\mu_2 = 1.50$ and $\lambda = 1$ μm. If the core radius is 50 μm, calculate the normalized frequency of the fibre (V) and the numb of guided modes. **(M.U. Dec. 2013) (5**

Solution :

Data : $\mu_1 = 1.53$, $\mu_2 = 1.50$ and $\lambda = 1\ \mu m = 10^{-6}$ m.

$a = 50\ \mu m = 5 \times 10^{-5}$ m.

Formula :

$$V = \frac{2\pi a}{\lambda}\sqrt{\mu_1^2 - \mu_2^2}$$

$$N_m = \frac{V^2}{2}$$

Calculations :

$$V = \frac{2 \times 3.14 \times 5 \times 10^{-5}}{2} \times \sqrt{(1.53)^2 - (1.5)^2}$$

$$\therefore \quad V = 94.71$$

$$N_m = \frac{V^2}{2} = \frac{(94.71)^2}{2} = 4484$$

Result :

$$V = 94.71$$

$$N_m = 4484$$

Problem 14

Calculate the V number of an optical fibre having numerical aperture 0.25 and co diameter 20 μm, if it is operated at 1.55 μm. **(M.U. May 2015; Dec. 2017) (3**

Solution :

Data : N.A. = 0.25, $a = 20\ \mu m = 2 \times 10^{-5}$ m

$\lambda = 1.55\ \mu m = 1.55 \times 10^{-6}$ m.

Formula :

$$V = \frac{2\pi a}{\lambda} \times N.A.$$

Calculations :

$$V = \frac{2 \times 3.14 \times 2 \times 10^{-5}}{1.55 \times 10^{-6}} \times 0.25 = 10.125$$

Result : V number = 10.125.

Problem 15

The core diameter of a multimode step index fibre is 50 μm. The numerical aperture is 0.25. Calculate the number of guided modes at an operating wavelength of 0.75 μm. **(M.U. Dec. 2015; May 2017) (3 m)**

Solution :

Data : $a = 50\ \mu m = 5 \times 10^{-5}$ m, N.A. = 0.25

$\lambda = 0.75\ \mu m = 7.5 \times 10^{-7}$ m.

Formula :

$$V = \frac{2\pi a}{\lambda} \times N.A.$$

$$N_m = \frac{V^2}{2}$$

Calculations :

$$V = \frac{2 \times 3.14 \times 5 \times 10^{-5}}{7.5 \times 10^{-7}} \times 0.25 = 52.36$$

$$N_m = \frac{V^2}{2} = \frac{(52.36)^2}{2} = 1370.$$

Result : Number of guided modes = 1370.

Problem 16

A step index fibre in air has N.A. of 0.16, a core refractive index 1.45 and a core diameter of 60 μm. Determine the normalized frequency for the fibre when light of wavelength of 0.9 μm is transmitted. **(M.U. Dec. 2016) (3 m)**

Solution :

Data : N.A. = 0.16, $\mu_1 = 1.45$, $d = 60 \times 10^{-6}$ m, $\lambda = 0.9 \times 10^{-6}$ m.

$$V = \frac{\pi d}{\lambda} NA$$

Formula :

Calculations : $$V = \frac{3.14 \times 60 \times 10^{-6}}{0.9 \times 10^{-6}} \times 0.16 = 33.49$$

Result : Normalized frequency, V = 33.49.

Problem 17

Calculate the number of modes an optical fibre of diameter 40 μm will trans[mit] its core and cladding refractive indices are 1.5 and 1.46 respectively. Wavelength of used is 1.5 μm. **(M.U. Dec. 2016; Nov. 2018)**

Solution :

Data : $d = 40\ \mu m = 40 \times 10^{-6}$ m, $\mu_1 = 1.5$, $\mu_2 = 1.46$,

$\lambda = 1.5\ \mu m = 1.5 \times 10^{-6}$ m.

Formula : $$V = \frac{\pi d}{\lambda}\sqrt{\mu_1^2 - \mu_2^2}, \quad N = \frac{V^2}{2} \text{ for SI fibre}$$

Calculations : $$V = \frac{3.14 \times 40 \times 10^{-6}}{1.5 \times 10^{-6}} \times \sqrt{(1.5)^2 - (1.46)^2} = 28.81$$

$$N = \frac{(28.81)^2}{2} = 415$$

Result : Number of modes, N = 415.

Problem 18

A graded index fibre has a core diameter of 0.05 mm and numerical aperture of [0.22] at a wavelength of 8500 A°. What is the normalized frequency and the number of m[odes] guided in the core?

Solution :

Data : NA = 0.22, $d = 0.05$ mm $= 5 \times 10^{-5}$ m,

$\lambda = 8500$ A° $= 8500 \times 10^{-10}$ m.

Formula : $$V = \frac{\pi d}{\lambda} NA, \quad N_m = \frac{V^2}{2}$$

Calculations : $$V = \frac{3.14 \times 5 \times 10^{-5}}{8500 \times 10^{-10}} \times 0.22 = 40.63$$

For G.I. Fibre, $$N_m = \frac{V^2}{4} = \frac{(40.63)^2}{4} = 412.5 \approx 412$$

Result : Normalized frequency, V = 40.63

Number of guided modes, $N_m = 412$.

Problem 19

The core diameter of a multimode step index fibre is 50 μm. The numerical aperture is 0.25. Calculate the number of guided modes at an operating wavelength of 0.75 μm. **(M.U. Nov. 2015; Dec. 2017) (3 m)**

Solution :

Data : $d = 50\ \mu m = 50 \times 10^{-6}$ m, NA = 0.25

$\lambda = 0.75\ \mu m = 75 \times 10^{-8}$ m.

Formula : $$N_m = \frac{V^2}{2} \text{ for MMSIF,} \quad V = \frac{\pi d}{\lambda} NA$$

Calculations : $$V = \frac{3.14 \times 50 \times 10^{-6}}{75 \times 10^{-8}} \times 0.25 = 52.36$$

$$N_m = \frac{V^2}{2} = \frac{(52.36)^2}{2} = 1370$$

Result : Number of guided modes = 1370.

Problem 20

Compute the maximum radius allowed for a fibre having core refractive index 1.47 and a cladding refractive index 1.46. The fibre is to support only one mode at a wavelength of 1300 nm. **(M.U. Dec. 2009; May 2013) (5 m)**

Solution :

Data : $\mu_1 = 1.47$, $\mu_2 = 1.46$,

$\lambda = 1300$ nm $= 1300 \times 10^{-9}$ m, $V_{max} = 2.405$ for SIF

Formula : $$V = \frac{2\pi r}{\lambda}\sqrt{\mu_1^2 - \mu_2^2}$$

Calculations : $$r_{max} = \frac{V_{max} \cdot \lambda}{2\pi\sqrt{\mu_1^2 - \mu_2^2}}$$

$$= \frac{2.405 \times 1300 \times 10^{-9}}{2 \times 3.14\sqrt{(1.47)^2 - (1.46)^2}}$$

$\therefore \quad r = 2.908 \ \mu m.$

Result : Radius, $r = 2.908 \ \mu m$.

Problem 21

A step index fibre has core diameter 29×10^{-6} m. The refractive indices of th[e core] and the cladding are 1.52 and 1.5189 respectively. If the light of wavelength 1.3 [μm is] transmitted through the fibre, determine :

(i) Normalized frequency of the fibre,

(ii) The number of modes the fibre will support. **(M.U. May 2013)**

Solution :

Data : $d = 29 \times 10^{-6}$ m, $\mu_1 = 1.52$, $\mu_2 = 1.5189$,

$\lambda = 1.3 \ \mu m = 1.3 \times 10^{-6}$ m.

Formula :
$$V = \frac{2\pi r}{\lambda}\sqrt{\mu_1^2 - \mu_2^2}$$

$$N_m = \frac{V^2}{2} \quad \text{for MMSIF } (\mu_1 = \text{constant})$$

Calculations :
$$V = \frac{3.14 \times 29 \times 10^{-6}}{1.3 \times 10^{-6}} \times \sqrt{(1.52)^2 - (1.5189)^2}$$

$\therefore \quad V = 4.049$

$$N_m = \frac{V^2}{2} = \frac{(4.049)^2}{2} = 8$$

Result : Normalized frequency, $V = 4.049$,

Number of modes, $N_m = 8$.

Problem 22

An optical fibre has core diameter of 6 μm and its core refractive index 1.45. T[he] **critical angle is 87°. Calculate** (i) refractive index of cladding, (ii) acceptance angle [and] **(iii) the number of modes propagating** through the fibre when wavelength of light is 1 [μm].

(M.U. May 2017)

Solution :

Data : $d = 6 \times 10^{-6}$ m, $\mu_1 = 1.45$, $\alpha_{min} = 87°$, $\lambda = 10^{-6}$ m.

Formula :
$$\alpha_{min} = \sin^{-1}\left(\frac{\mu_2}{\mu_1}\right), \quad i_{max} = \sin^{-1}\left(\sqrt{\mu_1^2 - \mu_2^2}\right)$$

$$V = \frac{\pi d}{\lambda}\sqrt{\mu_1^2 - \mu_2^2}$$

Calculations :
$$\mu_2 = \mu_1 \sin \alpha_{min} = 1.45 \sin 87° = 1.447$$

$$i_{max} = \sin^{-1}\left(\sqrt{(1.45)^2 - (1.447)^2}\right) = 0.0934°$$

$$V = \frac{3.14 \times 6 \times 10^{-6}}{10^{-6}} \times \left(\sqrt{(1.45)^2 - (1.447)^2}\right) = 1.756$$

Result : $\mu_2 = 1.447$, $i_{max} = 0.0934°$, $V = 1.756$.

Important Points to Remember

1. Snell's law of refraction
$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1}$$

2. Total internal reflection
$$\sin \alpha = \frac{\mu_2}{\mu_1}$$

3. Numerical aperture
$$NA = \mu_0 \sin i_{max} = \sqrt{\mu_1^2 - \mu_2^2}$$

4. Fractional refractive index change
$$\Delta = \frac{\mu_1 - \mu_2}{\mu_1}.$$

5. $$NA = \mu_1 \sqrt{2\Delta} \quad : \text{S.I. fibre}$$
$$= \mu_1 \sqrt{2\Delta}\sqrt{1 - (r/a)^2} \quad : \text{G.I. fibre}$$

6. Normalized frequency : V number.
$$V = \frac{2\pi a}{\lambda}\sqrt{\mu_1^2 - \mu_2^2}$$

$V \leq 2.405$: SMF

$V > 2.405$: MMF

7. Maximum number of allowed modes

$$N_m = \frac{V^2}{2} \quad \text{for SIF}$$

$$= \frac{V^2}{4} \quad \text{for GIF}$$

EXERCISE

1. Describe the propagation mechanism of light through an optical fibre.
2. Derive an expression for the numerical aperture for SI and GRIN fibre.
3. What is an optical fibre? Define and explain the terms
   (i) Numerical aperture (ii) Cone of acceptance
   (iii) Acceptance angle (iv) Relative refractive index difference
   (v) Propagating modes (vi) Normalizes frequency.
4. What are called modes of propagation? Discuss various types of modes.
5. Explain the allowed modes in optical fibres. How are they related to norma[l] frequency?
6. What are SMSIF, MMSIF and MMGIF? Compare their structural and transmiss[ion] characteristics.
7. Discuss the advantages if using optical fibres in communication systems
8. Differentiate between single mode and multimode fibres.

Problems for practice

1. The numerical aperture of an optical fibre is 0.5 and the core refractive index is 1[...] Find
   (i) the cladding refractive index, and
   (ii) the fractional index difference.
   [Ans. : 1.456, 0.0[...]
2. Find the core radius necessary for single mode transmission of a wavelength 8[...] in SI fibre with core refractive index 1.480 and cladding refractive index 1.47. W[...]

is the numerical aperture and the maximum acceptance angle of the fibre ?

[Ans. : 0.1717, 9° 53′ 12″]

Previous University Examination Questions with Solutions

1. Derive the expression of numerical aperture for a step index fibre. What is the importance of acceptance angle in fibre optics communication?
   [Refer § 2.11.1] **(M.U. Dec. 2002, 05, 07, 08, 11, 12, 15, 16; May 2013, 15) (5 m)**
2. Explain what are step index and graded index fibres. Explain their r.i. profiles.
   [Refer § 2.10.1] **(M.U. Dec 2003, 05, 10, 16; May 2011, 17) (4 m)**
3. Differentiate between SI fibre and GRIN fibre. Derive the expression for numerical aperature for both. **(M.U. Dec 2010, 14; May 2014, 18) (9 m)**
   [Refer § 2.10.1, 2.11.1, 2.11.3]
4. What is the difference between critical angle and acceptance angle?
   [Refer § 2.11.1] **(M.U. May 2010) (3 m)**
5. Define terms :
   (i) Total internal reflection
   (ii) Numerical aperture
   (iii) Acceptance angle **(M.U. May 2009; Dec. 2013, 14, 15) (3 m)**
   [Refer § 2.11.1]
6. Why a ray of light is zigzag in SI fibre and spiral in GRIN fibre?
   [Refer § 2.11.4] **(M.U. May 2012) (3 m)**
7. Distinguish between single mode and multimode fibres.
   [Refer Table 2.2] **(M.U. Dec 2009; May 2013) (3 m)**
8. What are monomode and multimode fibres? Explain V number.
   [Refer § 2.12] **(M.U. Dec. 2016; Nov. 2018) (3 m)**
9. What are the advantages of optical fibre? Explain the use of optical fibre in communication system. **(M.U. Dec. 2017) (7 m)**
   [Refer § 2.13]
10. State the advantages of optical fibre cables on conventional electrical cables.
    [Refer § 2.13] **(M.U. May 2019) (3 m)**

11. Why would you recommend the use of optical fibre in communication system? (M.U. May 2008, 16, 17; Dec. 2012) (3 M)

[Refer § 2.13]

12. Draw the block diagram of an optical fibre communication system and explain the function of each block. (M.U. Nov. 2018) (5 M)

[Refer § 2.13.1]

□□□

MODULE 3

Electrodynamics

(**Prerequisites :** Electric Charges, Coulomb's law-force between two point charges, Electric field, Electric field due to a point charge, Electric field lines, Electric dipole, Electric field due to a dipole, Gauss's law, Faraday's law.)

Scalar and Vector field, Physical significance of gradient, Curl and divergence in Cartesian co-ordinate system, Gauss's law for electrostatics, Gauss's law for magnetostatics, Faraday's Law and Ampere's circuital law, Maxwell's equations (Free space and time varying fields).

(03 Hours)

(Weightage - 12%)

Course Outcome : CO3 : Learner will be able to illustrate the fundamentals of electrodynamics with required mathematical concepts.

SYNOPSIS

3.1 Introduction

3.2 Prerequisites

3.3 Scalar and Vector Fields

3.4 Physical Significance of Gradient, Divergence and Curl

3.5 Maxwell's Equations : Gauss' law, Faraday's law and Ampere's Circutal law.

3.6 Solved Problems

Important Points to Remember

Exercise

Previous University Examination Questions with Solutions

3.1 Introduction

Electrodynamics is used to study the electric, magnetic and electroma... phenomena. To deal with various types of charge configurations, the study of diff... coordinate systems is essential. The static or time invariant electric and magnetic field... independent of each other. On the otherhand, the time varying electric and magnetic f... are interdependent. This gives rise to electromagnetic fields the foundation of which... set of four Maxwell's equations. Some important applications of electrodynamics... antenna, waveguides, satellites, etc.

3.2 Prerequisites

3.2.1 : Coordinate Systems

(a) The Cartesian or Rectangular Coordinate System

In Cartesian coordinate system, there are three coordinate axes perpendicular to e... other that intersect at the origin. It is customary to choose a right handed coordinate sy... as shown in Fig. 3.1. Here 'O' is the origin.

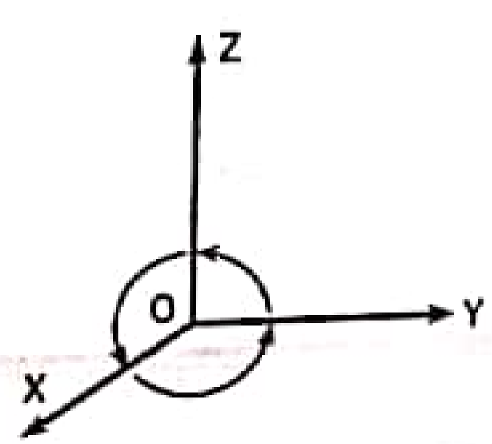


Fig. 3.1

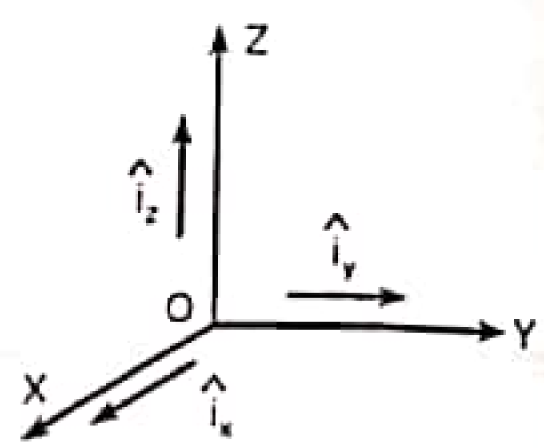


Fig. 3.2

(i) Unit Vectors

These are the vectors of unit length oriented strictly along the three axes of... coordinate system as shown in Fig. 3.2. These are denoted by $(\hat{i}_x, \hat{i}_y, \hat{i}_z)$ or $(\hat{i}, \hat{j}$... along X, Y, Z directions respectively. Being mutually perpendicular to each other the... vectors are related as follows :

$$\left.\begin{array}{l}\hat{i}_x \cdot \hat{i}_x = \hat{i}_y \cdot \hat{i}_y = \hat{i}_z \cdot \hat{i}_z = 1 \\ \hat{i}_x \times \hat{i}_y = \hat{i}_z, \quad \hat{i}_y \times \hat{i}_z = \hat{i}_x, \quad \hat{i}_z \times \hat{i}_x = \hat{i}_y\end{array}\right\} \quad \text{.............(}$$

(ii) Position Vectors

The *position vector* of P(x, y, z) [Fig. 3.3] and is written as

$$\vec{r} = x\,\hat{i}_x + y\,\hat{i}_y + z\,\hat{i}_z \quad \text{............... (3.2)}$$

The magnitude of which is

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2} \quad \text{............... (3.3)}$$

An unit vector $\hat{r}$ along the position vector $\vec{r}$ is written as

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} \quad \text{............... (3.4)}$$

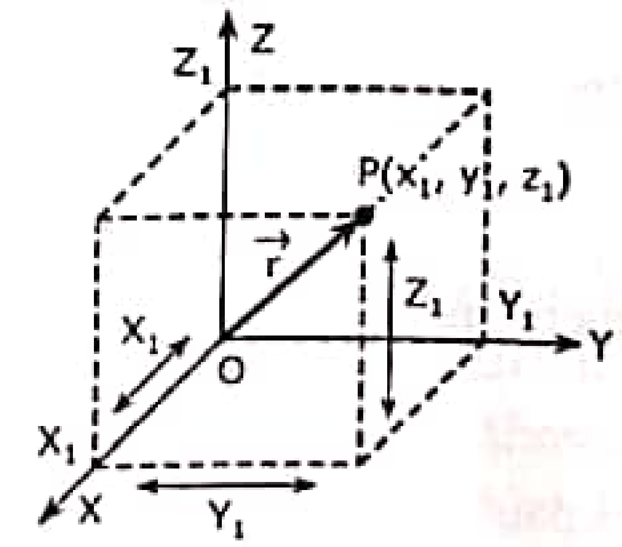


Fig. 3.3

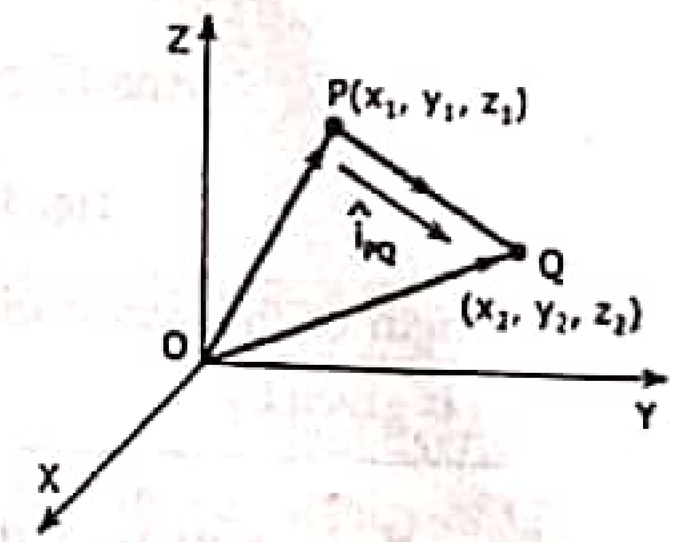


Fig. 3.4

(iii) Distance Vector : Displacement Vector

Consider a point $P(x_1, y_1, z_1)$ and a point $Q(x_2, y_2, z_2)$ as shown in Fig. 3.4.

The displacement vector or distance vector from a point $P(x_1, y_1, z_1)$ to point $Q(x_2, y_2, z_2)$ is represented by a vector $\vec{PQ}$ (Fig. 3.4), is given by

$$\vec{PQ} = (x_2 - x_1)\,\hat{i}_x + (y_2 - y_1)\,\hat{i}_y + (z_2 - z_1)\,\hat{i}_z \quad \text{............... (3.5)}$$

with magnitude

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad \text{............... (3.6)}$$

The unit vector along $\vec{PQ}$ is given by

$$\hat{i}_{PQ} = \frac{\vec{PQ}}{|\vec{PQ}|} \quad \text{............... (3.7)}$$

(iv) Differential Elements

Consider the point P(x, y, z). By increasing the coordinates, x, y and z infinitesi[mally] by dx, dy and dz respectively. The point reached is Q (x + dx, y + dy, z + dz). The incre[ments] dx, dy and dz from an infinitesimal box as shown in Fig. 3.5.

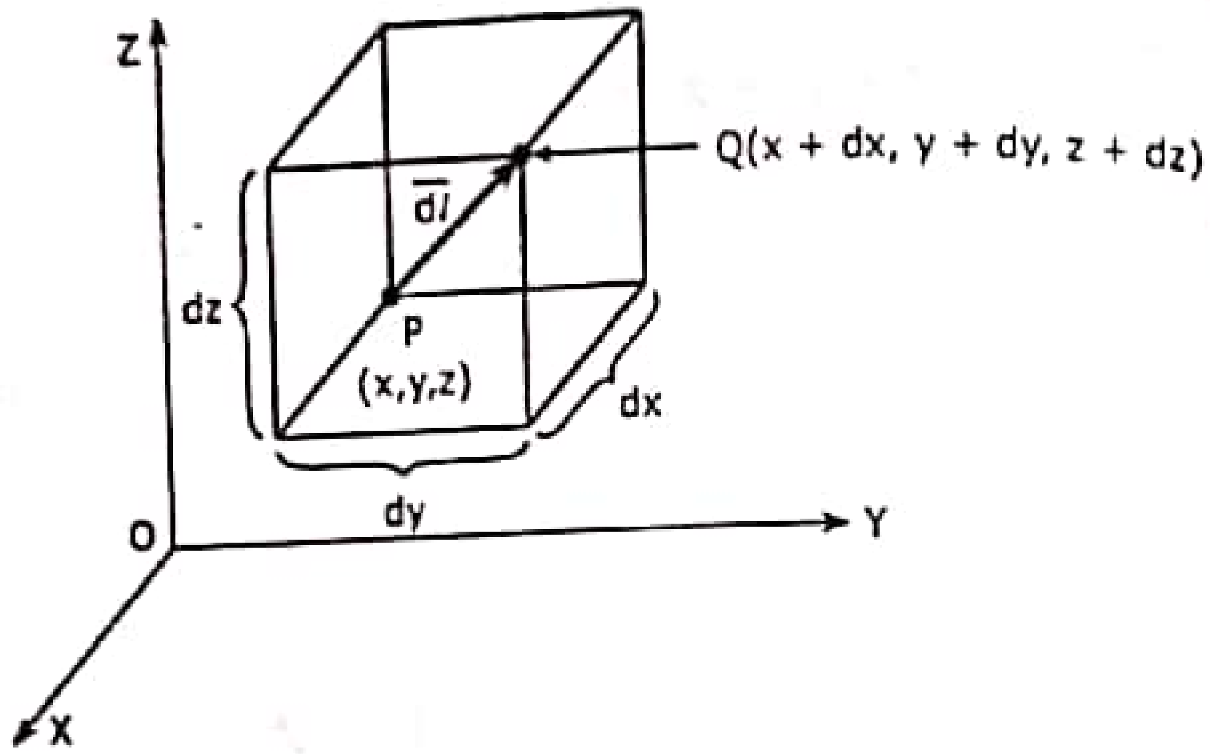


Fig. 3.5

The differential length or the vector displacement from P (x, y, z) to Q (x + dx, y + dy, z + dz) (Fig. 3.5) is given by

$$\boxed{\vec{dl} = dx\,\hat{i}_x + dy\,\hat{i}_y + dz\,\hat{i}_z}$$ ....

with magnitude of

$$|\vec{dl}| = \sqrt{dx^2 + dy^2 + dz^2}$$ ....

The three displacement components $dx\,\hat{i}_x$, $dy\,\hat{i}_y$ and $dz\,\hat{i}_z$ also define three surfa[ces] of infinitesimal areas in the three planes intersecting at P. The surface vector ha[s] magnitude, equal to the area and direction, normal to the area.

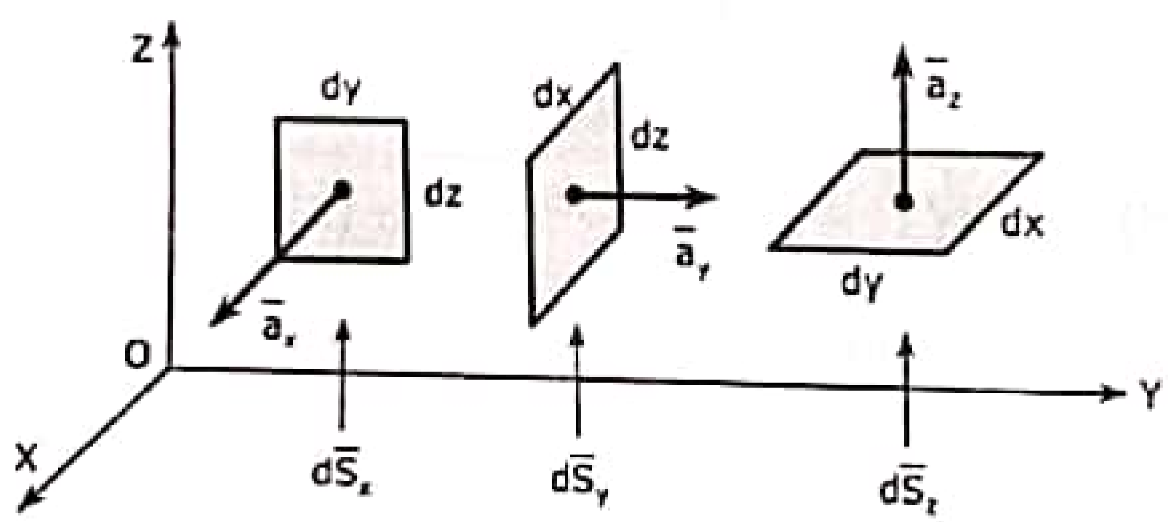


Fig. 3.6

The differential surface elements (as shown in Fig. 3.6) are thus given by

$$\left.\begin{aligned} \vec{dS}_x &= \pm\, dy\,dz\,\hat{i}_x \\ \vec{dS}_y &= \pm\, dz\,dx\,\hat{i}_y \\ \vec{dS}_z &= \pm\, dx\,dy\,\hat{i}_z \end{aligned}\right\} \quad \text{.............. (3.10)}$$

where ± sign takes into account two possible directions of normal to the surface.

The infinitesimal volume element is given by

$$\boxed{dv = dx\,dy\,dz} \quad \text{.............. (3.11)}$$

which is a scalar quantity.

3.2.2 : Fundamentals of Electromagnetic Theory

Coulomb's Law

If a static charge Q is placed in space, it develops a spherical electric field surrounding it. The lines of force of the field emanates radially outwards from Q as shown in Fig. 3.7. This field exerts a force on another charge Q' which is brought into the vicinity of Q. This force is given by Coluomb's law

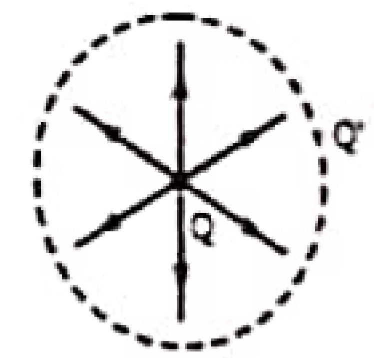


Fig. 3.7

$$\vec{F}_l = \frac{Q\,Q'}{4\pi\varepsilon_0 r^2}\,\hat{i}_r \quad (N) \quad \text{.............. (3.12)}$$

Here, $\varepsilon_0 = 8.854 \times 10^{-12}$ (C^2/Nm^2), permittivity of free space.

The Electric Field Intensity ($\vec{E}$)

Electric field intensity is defined as the force per unit charge at any point in the field region and is given by

$$\vec{E} = \frac{Q}{4\pi\epsilon_0\, r^2}\,\hat{i}_r \quad (N/C) \quad \text{.............. (3.13)}$$

The number of lines of force passing through unit surface area.

Electric Flux Density (ϕ) : Electric Displacement ($\vec{D}$)

Electric flux density is generally defined as the number of lines of force passing through unit surface area of the field region. On the other hand, electric displacement is defined as the electric charge over unit area of the spherical surface with its centre at

charge Q. Thus, electric displacement is same as electric charge density and is wr...

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{i}_r = \varepsilon_0 \vec{E} \quad \text{.......... (3.}$$

Total flux over the complete spherical surface is given by

$$\phi = \int_S \vec{D} \cdot \vec{ds} \quad \text{.......... (3.}$$

Electric Potential

The work done by an external source in moving a charge Q from one point to ano... in an electric field $\vec{E}$ is

$$V_{AB} = \frac{W}{Q} = -\int_B^A \vec{E} \cdot \vec{dl} \quad \text{.......... (3.}$$

...s seen in Fig. 3.8.

This is called the ***potential difference*** between points A and B and is given by

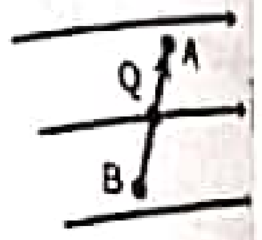


Fig. 3.8

$$V_{AB} = -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_B^A$$

$$\therefore \quad V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \quad \text{.............. (3.17)}$$

Here, r_A and r_B are the position vectors of points A and B.

The ***absolute potential*** at a point P at a distance R from a charge Q as shown in Fig. 3.9 is given by

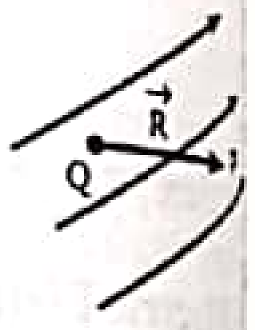


Fig. 3.9

$$V = \frac{Q}{4\pi\varepsilon_0 R} \quad \text{.............. (3.18)}$$

The electrical field at any point in the field is given by

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{in scalar form} \quad \text{.......... (3.}$$

$$\text{and} \quad \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{i}_r \quad \text{in vector form} \quad \text{.......... (3.}$$

Magnetic Field

The region around a magnet within which the influence of the magnet can be experienced is called the magnetic field. The magnetic lines of force or magnetic flux lines start from the north pole and end of the south pole.

An isolated magnetic pole can never exist.

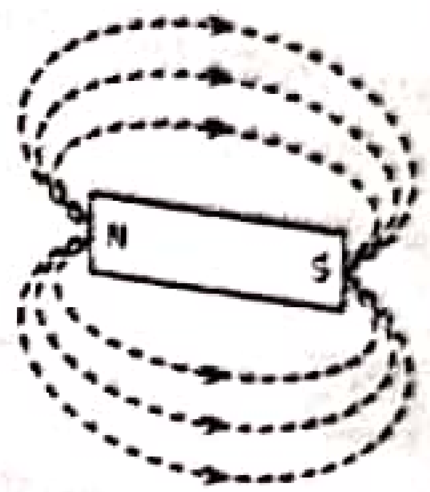


Fig. 3.10

Magnetic Field Intensity ($\vec{H}$)

The magnetic field intensity at any point in the magnetic field is defined as the force experienced by a unit north pole of one weber strength, when placed at that point.

The magnetic flied intensity is measured in (N/wb). This is similar to the electric field intensity, $\vec{E}$ in electrostatics.

Magnetic Flux Density ($\vec{B}$)

The total magnetic lines of force i.e., magnetic flux crossing a unit surface area in a perpendicular direction is called magnetic flux density. This is measured in Wb/m^2 and is very similar to the electric flux density $\vec{D}$ in electrostatics.

The magnetic field intensity ($\vec{H}$) and the magnetic flux density ($\vec{B}$) are related as

$$\vec{B} = \mu \vec{H} \quad \text{.............. (3.21)}$$

where, $\mu = \mu_0 \mu_r$; permeability of the medium,

with $\mu_0 = 4\pi \times 10^{-7}$ (N / A^2), permeability of free space,

and μ_r = relative permeability.

3.3 Scalar and Vector Fields

Field : The behaviour of a physical quantity in a given region is described by its value at each point in that region. ***A field is a function that describes the behaviour of a physical quantity at all points in a given region of space.*** The physical quantity described by the field can be either a scalar or a vector. Thus a field can also be a scalar field or a vector field.

Scalar field : A scalar field is specified by the magnitude of a physical quantity at each point of the field region. Some examples of scalar fields are temperature, pressure, electric potential, etc.

Vector field : A vector field is specified by both the magnitude and the direction of a physical quantity at each point of the field region. Examples of vector fields are velocity, acceleration, electric field, etc.

3.4 Physical Significance of Gradient, Divergence and Curl

3.4.1 : The Del Operator : $\vec{\nabla}$

We introduce a vector differential operator which is essential for the study of electrodynamics.

The del operator expressed in Cartesian coordinates, as

$$\vec{\nabla} = \hat{i}_x \frac{\partial}{\partial x} + \hat{i}_y \frac{\partial}{\partial y} + \hat{i}_z \frac{\partial}{\partial z} \quad \text{.............. (3}$$

3.4.2 : Gradient of a Scalar Field

Gradient is a mathematical operation performed on a scalar field which results in a vector field. Gradient is a vector that represents both the magnitude and the direction of maximum space rate of increase of a scalar.

If V = V (x, y, z) is a scalar function the gradient operation is written in Cartesian coordinates, as

$$\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{i}_x + \frac{\partial V}{\partial y}\hat{i}_y + \frac{\partial V}{\partial z}\hat{i}_z \quad \text{.............. (3}$$

Gradient is the multidimensional rate of change of a given function.

At any point in the scalar field.

(i) the magnitude of the resulting vector field is the maximum rate of increase of the scalar field.

(ii) the direction of the resulting vector field is the direction in which the maximum rate of increase occurs.

3.4.3 : Divergence of a Vector Field

The rate of change of a vector field is complex. *The divergence of a vector field indicates how much the vector field spreads out from a certain point.*

Imagine a fluid, with the vector field representing the velocity of the fluid at each point in space. Divergence measures the net flow of the fluid out of a given point. If the fluid is flowing into that point, the divergence will be negative. *The divergence of a vector is a scalar.*

In Cartesian coordinates, $\vec{B}$ is a vector field given by

$$\vec{B} = B_x \hat{i}_x + B_y \hat{i}_y + B_z \hat{i}_z \quad \text{.............. (3.24)}$$

and its divergence is written as

$$\vec{\nabla} \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \quad \text{.............. (3.25)}$$

3.4.4 : Curl of a Vector Field

The curl of a vector field at any point signifies how much the vector quantity curls or twists around that point. As an example, we may consider water going down the drain. In this motion water swirls in rotation. The curl of the velocity field of water describes its local rotation. The rotation has a direction and is about the direction of motion. The curl of a vector field is a vector field and is written as $\vec{\nabla} \times \vec{B}$.

Mathematical expression for the curl $\vec{B}$ in Cartesian coordinates can be obtained by solving the determinant as given below.

For $$\vec{B} = B_x \hat{i}_x + B_y \hat{i}_y + B_z \hat{i}_z \quad \text{.............. (3.26)}$$

$$\text{Curl } \vec{B} = \vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{B} = \hat{i}_x \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \hat{i}_y \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + \hat{i}_z \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \quad \text{.............. (3.27)}$$

If the curl of a vector field is zero the filed is irrotational and is called a conservative field.

3.4.5 : Divergence of a Curl is Zero

For any field $\vec{B}$ the divergence of a curl of $\vec{B}$ is written as

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \left(\hat{i}_x\frac{\partial}{\partial x}+\hat{i}_y\frac{\partial}{\partial y}+\hat{i}_z\frac{\partial}{\partial z}\right)\cdot$$

$$\left[\hat{i}_x\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)+\hat{i}_y\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)+\hat{i}_z\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\right]$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)$$

$$=0$$

3.4.6 : Divergence Theorem and Stokes Theorem

These are two important theorems essential for the study of electromagnetism.

(A) Divergence Theorem

The volume integral of the divergence of a vector field $\vec{A}$ taken over any volume is equal to the surface integral of $\vec{A}$ taken over the surface enclosing the volume v. The

$$\int_v \vec{\nabla}\cdot\vec{A}\,dv = \oint_S \vec{A}\cdot\vec{ds} \quad \text{.............. (3.}$$

Here the direction of $\vec{ds}$ is always outward normal as shown in *Fig. 3.11.*

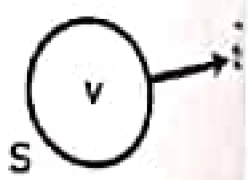


Fig. 3.11

The divergence theorem is used to convert a volume integral to a surface integral and vice-versa.

(B) Stokes' Theorem

The surface integral of the curl of a vector field $\vec{A}$ over an open surface is equal to the line integral of the vector field over the closed curve bounding the surface area. That is

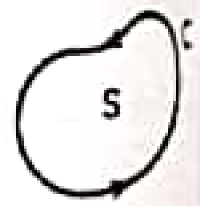


Fig. 3.12

$$\int_S (\vec{\nabla}\times\vec{A})\cdot\vec{ds} = \oint_c \vec{A}\cdot\vec{dl} \quad \text{.............. (3.}$$

The Stokes' theorem is used to replace a surface integral by a line integral or vice versa.

3.5 Maxwell's Equations

Maxwell's equations, a set of four equations form the foundation of electromagnetic theory. These are extensions of works of Gauss, Faraday and Ampere.

Maxwell's equations are classified into following categories -

(i) For static fields and for time varying fields.

(ii) In differential form and in integral form.

3.5.1 : Static and Time Varying Fields

Static fields : If the value of the physical quantity describing the field does not vary with time the field is called time invariant or a static field.

Both static electric and magnetic fields are used in the design of many devices. For example, a static electric field can accelerate an electro and a static magnetic field can deflect it, this scheme is employed in the design of an oscilloscope and an ink-jet printer.

Time varying field : If the value of the physical quantity changes with time in the field region the field is a time varying field. ***Time varying electric and magnetic fields are coupled resulting in an electromagnetic field.***

3.5.2 : Laws for Static Electric and Magnetic Fields

(A) Gauss' Law for Electric Field

Gauss' law : The electric flux passing through any closed surface is equal to the total charge enclosed by that surface. This is mathematically stated as "the surface integral of the normal component of electric flux density $\vec{D}$ over any closed surface equals the charge enclosed" and is written as

$$\phi = \oint_S \vec{D}\cdot\vec{ds} = Q_{enclosed} \quad \text{........ (3.30)}$$

Here, Q being the total charge enclosed by the closed surface as shown in Fig. 3.13. This may be expressed as the volume integral of the charge density ρ_v. So Gauss' law is written in ***integral form*** as

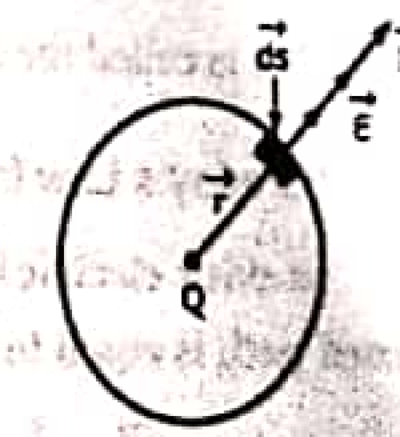


Fig. 3.13

$$\oint_S \vec{D}\cdot\vec{ds} = \int_v \rho_v\,dv \quad \text{.............. (3.31)}$$

Applying divergence theorem, the surface integral is converted to a volume in... and we write

$$\oint_S \vec{D} \cdot \vec{ds} = \int_v (\vec{\nabla} \cdot \vec{D}) dv \quad \text{.........(3.}$$

Hence, equation (3.31) becomes

$$\int_v (\vec{\nabla} \cdot \vec{D}) dv = \int_v \rho_v \, dv \quad \text{.........(3.}$$

Hence, Gauss' law for electric field in *differential form or point form* is written

$$\vec{\nabla} \cdot \vec{D} = \rho_v \quad \text{.........(3.}$$

In an electric or a magnetic field, any closed surface, real or imaginary, is called a Gaussian surface.

(B) Gauss' Law for Static Magnetic Field

In a magnetic field the magnetic lines are closed on themselves as seen in Fig. 3... **Hence, the total outgoing magnetic flux is zero.** This is written as

$$\oint_S \vec{B} \cdot \vec{ds} = 0 \quad \text{.........(3.}$$

and is called the *Gauss' law* for magnetic field in integral form.

Using divergence theorem the magnetic Gauss' law can be written as

$$\oint_S \vec{B} \cdot \vec{ds} = \oint_S (\vec{\nabla} \cdot \vec{B}) dv = 0 \quad \text{.........(3.}$$

$$\text{or} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \text{.........(3.}$$

This is called the *differential or point form of magnetic Gauss' law.*

(C) Faraday's Law for Static Electric Field

In static electric fields the work done involved in moving a test charge around closed path is equal to zero. Such fields are called *conservative fields*. In this case

$$\oint_C \vec{E} \cdot \vec{dl} = 0 \quad \text{.........(3.}$$

This is *integral form of Faraday's law* for static electric field. Using Stoke's theorem this can be written as

$$\oint_C \vec{E} \cdot \vec{dl} = \oint_S (\vec{\nabla} \times \vec{E}) \cdot \vec{ds} = 0 \quad \text{.........(3.39)}$$

$$\text{Hence,} \quad \vec{\nabla} \times \vec{E} = 0 \quad \text{.........(3.40)}$$

This is *differential or point form of Faraday's law* for static electric field.

(D) Ampere's Circuital Law for Static Magnetic Field

Ampere's circuital law states that "the line integral of magnetic field intensity H around a closed path is exactly equal to the direct current enclosed by that path". The mathematical representation of Ampere's law is

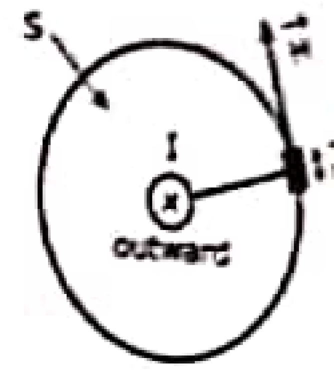


Fig. 3.14

$$\oint \vec{H} \cdot \vec{dl} = I \quad \text{.........(3.41)}$$

The law is very useful to determine $\vec{H}$ when the current distribution is symmetrical.

Since $\vec{B} = \mu \vec{H}$, equation (3.38) can be written as

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I \quad \text{.........(3.42)}$$

This is called the *integral form of Ampere's circuital law.*

Replacing $I = \int_S \vec{J} \cdot \vec{ds}$ where $\vec{J}$ is the current density and S is the surface area bounded by the path of integration of $\vec{H}$, we can write

$$\oint \vec{H} \cdot \vec{dl} = \int_S \vec{J} \cdot \vec{ds} \quad \text{.........(3.43)}$$

Using Stoke's theorem, this can be rewritten as

$$\oint_S (\vec{\nabla} \times \vec{H}) \cdot \vec{ds} = \int_S \vec{J} \cdot \vec{ds} \quad \text{.........(3.44)}$$

$$\text{Hence,} \quad \vec{\nabla} \times \vec{H} = \vec{J} \quad \text{.........(3.45)}$$

Since, $\vec{B} = \mu_0 \vec{H}$ in free space, we can write

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \text{.........(3.46)}$$

This is called the *differential or point form of Ampere's circuital law.*

(E) Equation of Continuity

Consider a small volume element Δv as shown in Fig. 3.15 located inside a conducting medium. The current density has the direction of current flow.

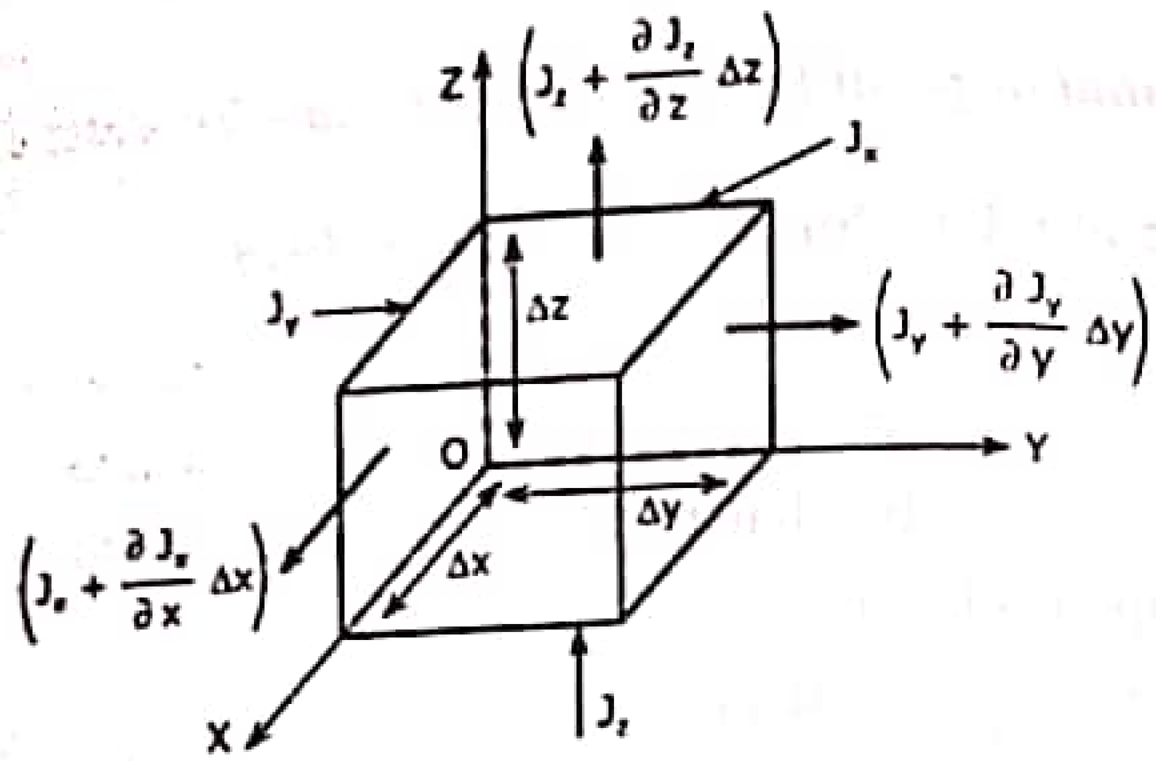


Fig. 3.15

If there is no source or sink of charge inside the volume $\Delta v\ (= \Delta x\ \Delta y\ \Delta z)$, the current is steady and continuous and so is current density as shown in Fig. 3.20. We have

$$\vec{\nabla} \cdot \vec{J} = 0$$

or

$$\oint_S \vec{J} \cdot \vec{ds} = 0 \qquad \text{.............. (3.4}$$

All three components and their variation are shown in the Fig. 3.15.

Here $\frac{\partial J_x}{\partial x}$, $\frac{\partial J_y}{\partial y}$, and $\frac{\partial J_z}{\partial z}$ are the rate of change of J_x, J_y and J_z in x, y and z directions respectively.

If the current is not steady, the difference between the current flowing into the volume and that flowing out of the volume must equal the rate of change of electric charge inside the volume.

A net flow of current out of the volume (positive current flow) must be equal to the negative rate of change of charge with time (rate of decrease of charge) within the volume. This is expressed by the continuity equation,

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \qquad \text{.............. (3.4}$$

where ρ_v is the volume charge density.

This is the general relation between current density $\vec{J}$ and the volume charge density ρ_v at a point.

3.5.3 : Fundamental Postulates of Electrostatics and Magnetostatics

The static electric and magnetic fields are governed by the following postulates that form the foundation of electrostatics and magnetostatics.

Table 3.1

Differential form	Integral form	Significance
$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$	$\int_S \vec{E} \cdot \vec{ds} = \frac{Q}{\varepsilon_0}$	Gauss's law for electrostics
$\vec{\nabla} \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot \vec{ds} = 0$	Gauss's law for magnetostatics
$\vec{\nabla} \times \vec{E} = 0$	$\oint_C \vec{E} \cdot \vec{dl} = 0$	Faraday's law of electrostatics
$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$	$\oint_C \vec{B} \cdot \vec{dl} = \mu_0 I$	Ampere's circulal law of magnetostatics

3.5.4 : Time Varying Electric and Magnetic Fields

In time varying electric and magnetic fields, the Faraday's law and Ampere's circuital law are modified as follows :

(A) Faraday's Law in Time Varying Fields

A time varying magnetic field produces an electro motive force (emf) which may establish a current in a suitable closed circuit.

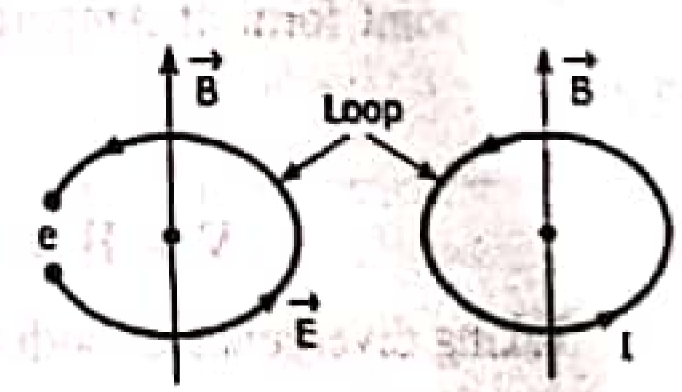


Fig. 3.16

Faraday's law in general is stated as

$$\text{emf} = e = -\frac{\partial \phi}{\partial t} \qquad \text{.............. (3.49)}$$

The emf induced in the loop (Fig. 3.16) e is given by

$$e = \vec{E} \cdot \vec{dl} \qquad \text{.............. (3.50)}$$

where $\vec{E}$ is the emf producing field.

The total flux through the circuit is given by

$$\phi = \oint_S \vec{B} \cdot \vec{ds} \quad \text{.............. (3.51)}$$

Substituting equations (3.50) and (3.51) in equation (3.49), it is obtained that

$$\oint \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t} \oint_S \vec{B} \cdot \vec{ds}$$

$$\oint \vec{E} \cdot \vec{dl} = -\oint_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} \quad \text{.............. (3.52)}$$

This is the ***integral form of Faraday's law.***

By Stoke's theorem, equation (3.52) can be written as

$$\int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{ds} = -\oint_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

It follows, $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{.............. (3.53)}$$

This is the ***differential or point form of Faraday's law.***

3) Modification of Ampere's Law in Time Varying Fields

Inconsistency in Ampere's law derived for static fields :

The point form of Ampere's law derived for magnetostatics is given in equation (3.45) as

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

Taking divergence of both sides of it we get

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J} \quad \text{.............. (3.54)}$$

Since divergence of a curl is zero,

$$\vec{\nabla} \cdot \vec{J} = 0 \quad \text{.............. (3.55)}$$

which is not consistent with the continuity equation (3.48).

Hence, statement of Ampere's law is inconsistent and requires some modifications.

Let us add an unknown variable $\vec{G}$ to equation (3.54), the Ampere's law becomes

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{G} \quad \text{.............. (3.56)}$$

Taking divergence of both sides, we get

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot (\vec{J} + \vec{G})$$

i.e., $$\vec{\nabla} \cdot (\vec{J} + \vec{G}) = 0 \quad \text{.............. (3.57)}$$

Since divergence of a curl is zero,

i.e., $$\vec{\nabla} \cdot \vec{J} = -\vec{\nabla} \cdot \vec{G} \quad \text{.............. (3.58)}$$

Using continuity equation here, we get

$$\vec{\nabla} \cdot \vec{G} = \frac{\partial \rho_v}{\partial t} \quad \text{.............. (3.59)}$$

The point or differential form of Gauss' law is

$$\vec{\nabla} \cdot \vec{D} = \rho_v$$

Hence, $$\vec{\nabla} \cdot \vec{G} = \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot \frac{\partial \vec{D}}{\partial t}$$

So we obtain, $$\vec{G} = \frac{\partial \vec{D}}{\partial t} \quad \text{.............. (3.60)}$$

These ***Ampere's law in differential or point form*** becomes

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}} \quad \text{.............. (3.61)}$$

Integrating it over a surface area S, we get

$$\int_S (\vec{\nabla} \times \vec{H}) \cdot \vec{ds} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{ds}$$

Applying Stoke's theorem, we obtain

$$\boxed{\oint \vec{H} \cdot \vec{dl} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{ds}} \quad \text{.............. (3.62)}$$

the ***integral form of Ampere's law.***

3.5.5 : Maxwell's Equations : General Form

The general set of four Maxwell's equations for time varying electromagnetic fields are listed below.

Table 3.2

Differential (Point) form	Integral form	Significance
$\vec{\nabla} \cdot \vec{D} = \rho$	$\oint \vec{D} \cdot \vec{ds} = \int_V \rho \, dv$	Gauss's law for electrostatics
$\vec{\nabla} \cdot \vec{B} = 0$	$\oint \vec{B} \cdot \vec{ds} = 0$	Gauss's law for magnetostatics (non-existance of magnetic monopole)
$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}$	$\oint \vec{E} \cdot \vec{dl} = -\int_S \dot{\vec{B}} \cdot \vec{ds}$	Faraday's law
$\vec{\nabla} \times \vec{H} = \vec{J} + \dot{\vec{D}}$	$\oint \vec{H} \cdot \vec{dl} = \int_S (\vec{J} + \dot{\vec{D}}) \cdot \vec{ds}$	Ampere's law
Supplementary equation		
$\vec{\nabla} \cdot \vec{J} = -\dot{\rho}$	$\oint_S \vec{J} \cdot \vec{ds} = -\int_V \rho \, dv$	Continuity equation

Maxwell's Equations in Free Space

In free space, there is no charge and no current. Hence, $\rho = 0$ and $\vec{J} = 0$ and Maxwell equations are listed as given below.

Table 3.3

Differential (Point) form	Integral form	Significance
$\vec{\nabla} \cdot \vec{D} = 0$	$\oint \vec{D} \cdot \vec{ds} = 0$	Gauss's law for electrostatics
$\vec{\nabla} \cdot \vec{B} = 0$	$\oint \vec{B} \cdot \vec{ds} = 0$	Gauss's law for magnetostatics (non-existance of magnetic monopole)

Table 3.3 contd.

$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}$	$\oint \vec{E} \cdot \vec{dl} = -\int_S \dot{\vec{B}} \cdot \vec{ds}$	Faraday's law
$\vec{\nabla} \times \vec{H} = \vec{J} + \dot{\vec{D}}$	$\oint \vec{H} \cdot \vec{dl} = \int_S \dot{\vec{D}} \cdot \vec{ds}$	Ampere's law

3.6 Solved Problems

Problem 1

A vector field $\vec{W}$ is given in Cartesian coordinate system as

$$\vec{W} = (4x^2y)\,\hat{i}_x - (7x+2z)\,\hat{i}_y + (4xy+2z^2)\,\hat{i}_z$$

(i) Calculate the magnitude of $\vec{W}$ at point P (2, – 3, 4).

(ii) Obtain a unit vector that shows the direction of the vector field at point P.

Solution :

(i) At point P (2, – 3, 4), x = 2, y = – 3, z = 4 and

$$\vec{W} = -48\,\hat{i}_x - 22\,\hat{i}_y + 8\,\hat{i}_z$$

$$|\vec{W}| = \sqrt{(-98)^2 + (-22)^2 + 8^2} = 53.4041$$

(ii) Unit vector in the direction of $\vec{W}$ at point P is

$$\hat{i}_P = \frac{\vec{W} \text{ at P}}{|\vec{W}| \text{ at P}} = \frac{-48\,\hat{i}_x - 22\,\hat{i}_y + 8\,\hat{i}_z}{53.4041}$$

$$\therefore \quad \hat{i}_P = -0.8988\,\hat{i}_x - 0.4119\,\hat{i}_y + 0.1498\,\hat{i}_z$$

Problem 2

Find the divergence of the vector field $\vec{F} = x^2yz\,\hat{i}_x + xz\,\hat{i}_y$ in Cartesian coordinates.

(M.U. May 2018) (5 m)

Solution :

$$\text{Divergence, } \vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{i}_x + \frac{\partial}{\partial y}\hat{i}_y + \frac{\partial}{\partial z}\hat{i}_z\right) \cdot \left(x^2yz\,\hat{i}_x + xz\,\hat{i}_y\right)$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xz) = 2xyz$$

Result : $\vec{\nabla}\cdot\vec{F} = 2xyz$

Problem 3

Find the divergence of vector function $\vec{A} = x^2\,\hat{i} + x^2y^2\,\hat{j} + 24\,x^2y^2z^3\,\hat{k}$.

(M.U. Nov. 2018)

Solution :

Divergence of $\vec{A} = \vec{\nabla}\cdot\vec{A}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x^2\,\hat{i} + x^2y^2\,\hat{j} + 24\,x^2y^2z^3\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(x^2y^2) + \frac{\partial}{\partial z}(24\,x^2y^2z^3)$$

$$= 2x + 2xy^2 + 72\,x^2y^2z^2$$

Result : $\vec{\nabla}\cdot\vec{A} = 2x\,(1 + y^2 + 36\,x\,y^2z^2)$

Problem 4

Find the divergence of the vector field $\vec{F} = x^2y\,\hat{i} - (z^3 - 3x)\,\hat{j} + 4y^2\,\hat{k}$.

(M.U. May 2019)

Solution :

Divergence of $\vec{F} = \vec{\nabla}\cdot\vec{F}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left[x^2y\,\hat{i} - (z^3 - 3x)\,\hat{j} + 4y^2\,\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(z^3 - 3x) + \frac{\partial}{\partial z}(4y^2)$$

$$= 2xy - 0 - 0 = 2xy$$

Result : $\vec{\nabla}\cdot\vec{F} = 2xy.$

Problem 5

If $\vec{A} = x^2z\,\hat{i} - 2\,y^2z^2\,\hat{j} + x\,y^2z\,\hat{k}$, find $\vec{\nabla}\cdot\vec{A}$ at point (1, – 1, 1).

(M.U. May 2017)

Solution :

$$\vec{\nabla}\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x^2z\,\hat{i} - 2\,y^2z^2\,\hat{j} + x\,y^2z\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2z) - \frac{\partial}{\partial y}(2y^2z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 4yz^2 + xy^2$$

At point (1, – 1, 1), $\vec{\nabla}\cdot\vec{A} = 7.$

Problem 6

Calculate the area of a rectangular surface of dimensions -2 cm $\le x \le 2$ cm, 1 cm $\le y \le 5$ cm and $z = 2$ cm.

Solution :

The surface ABCD has four corners,

A (– 2, 1, 2), B (– 2, 5, 2), C (2, 5, 2), D (2, 1, 2).

A surface element in ABCD is

$$ds = dx\,dy\,\hat{i}_z$$

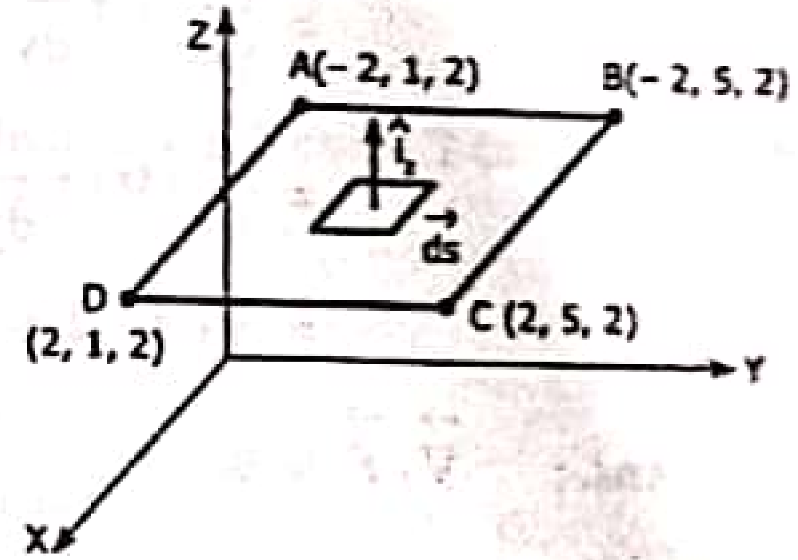


Figure for problem 6

Total area

$$\vec{S} = \int_{x=-2}^{2}\int_{y=1}^{5} dx\,dy\,\hat{i}_z$$

$$= [x]_{-2}^{2}\,[y]_1^5\,\hat{i}_z$$

and $\vec{S} = 16\,\hat{i}_z$

Result : Area is $16\,\hat{i}_z$ (Cm²).

Problem 7

If a scalar field $\phi = 3\,(x^2y - y^2x)$, calculate its gradient at the point (1, – 2, – 1).

Solution :

$$\text{Grad}\ \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot 3(x^2y - y^2x)$$

$$= 3\left[\hat{i}\frac{\partial}{\partial x}(x^2y - y^2x) + \hat{j}\frac{\partial}{\partial y}(x^2y - y^2x) + \hat{k}\frac{\partial}{\partial z}(x^2y - y^2x)\right]$$

$$= 3\left[\hat{i}(2xy - y^2) + \hat{j}(x^2 - 2yx) + 0\right]$$

$$\text{Grad } \phi = 3\left[\hat{i}\,y(2x - y) + \hat{j}\,x(x - 2y)\right]$$

At point (1, – 2, – 1), this becomes,

$$\text{Grad } \phi = -24\,\hat{i} + 15\,\hat{j}.$$

Problem 8

Calculate the divergence of the vector field $\vec{F} = 3x\,\hat{i} + 4y\,\hat{j} + 2z\,\hat{k}$.

Solution :

$$\text{Divergence of } \vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(3x\,\hat{i} + 4y\,\hat{j} + 2z\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(2z)$$

$$= 3 + 4 + 2 = 9$$

Ans. : $\vec{\nabla}\cdot\vec{F} = 9$

Problem 9

A vector field is given as $\vec{F} = y\,\hat{i} + (x^2 + y^2)\,\hat{j} + (yz + zx)\,\hat{k}$. Find (i) Div. $\vec{F}$, (ii) Curl $\vec{F}$.

Solution :

(i) $$\text{Div. } \vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(y\,\hat{i} + (x^2 + y^2)\,\hat{j} + (yz + zx)\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x^2 + y^2) + \frac{\partial}{\partial z}(yz + zx)$$

$$\therefore \text{Div. } \vec{F} = 0 + 2y + (y + x) = x + 3y$$

(ii) $$\text{Curl } \vec{F} = \vec{\nabla}\times\vec{F}$$

$$= \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x^2 + y^2 & yz + zx \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(yz + zx) - \frac{\partial}{\partial z}(x^2 + y^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(y) - \frac{\partial}{\partial x}(yz + zx)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(y)\right]$$

$$= \hat{i}\,z + \hat{j}(-z) + \hat{k}(2x - 1)$$

$$\therefore \text{Curl } \vec{F} = z\,\hat{i} - z\,\hat{j} + (2x - 1)\,\hat{k}$$

Problem 10

Find the volume of a block bounded by 1 cm ≤ x ≤ 3 cm, – 2 cm ≤ y ≤ 4 cm, – 1 cm ≤ z ≤ 3 cm.

Solution :

$$\text{The volume is } V = \int_3 dV = \int_{x=1}^{3}\int_{y=-2}^{4}\int_{z=-1}^{3} dx\,dy\,dz$$

$$= \int_1^3 dx \int_{-2}^4 dy \int_{-1}^3 dz = [x]_1^3\,[y]_{-2}^4\,[z]_{-1}^3$$

$$= 2 \times 6 \times 4 = 48 \text{ cm}^3$$

Problem 11

Find the divergence and curl of the field $\vec{F} = 30\hat{i}_x + 2xy\,\hat{i}_y + 5xz^2\,\hat{i}_z$ in Cartesian coordinates.

Solution :

$$\text{Divergence, } \vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i}_x + \frac{\partial}{\partial y}\hat{i}_y + \frac{\partial}{\partial z}\hat{i}_z\right)\cdot\left(30\hat{i}_x + 2xy\,\hat{i}_y + 5xz^2\,\hat{i}_z\right)$$

$$= \frac{\partial}{\partial x}(30) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(2xz^2) = 2x + 10xz$$

$$\vec{\nabla}\cdot\vec{F} = 2x(1+5z)$$

$$\text{Curl, } \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 30 & 2xy & 5xz^2 \end{vmatrix} = -5z^2\,\hat{i}_y + 2y\,\hat{i}_z$$

Results :

$$\vec{\nabla}\cdot\vec{F} = 2x(1+5z)$$

$$\vec{\nabla}\times\vec{F} = -5z^2\,\hat{i}_y + 2y\,\hat{i}_z$$

Problem 12

Given, $\vec{A} = y\cos ax\,\hat{i}_x + (y+e^x)\,\hat{i}_z$. Find curl $\vec{A}$ at the origin.

Solution :

In Cartesian coordinates

$$\text{Curl A} = \nabla\times\vec{A} = \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

In this case, $A_x = y\cos ax$, $A_y = 0$, $A_z = y + e^x$.

$$\text{Curl } \vec{A} = \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & 0 & A_z \end{vmatrix}$$

$$= \frac{\partial A_z}{\partial y}\hat{i}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{i}_y - \frac{\partial A_x}{\partial y}\hat{i}_z$$

$$\text{Curl } \vec{A} = \frac{\partial}{\partial y}(y+e^x)\,\hat{i}_x + \left[\frac{\partial}{\partial z}(y\cos ax) - \frac{\partial}{\partial x}(y+e^x)\right]\hat{i}_y - \frac{\partial}{\partial y}(y\cos ax)\,\hat{i}_z$$

$$\text{Curl } \vec{A} = \hat{i}_x + [0 - e^x]\,\hat{i}_y - \cos ax\,\hat{i}_z$$

$$\therefore \quad \nabla\times\vec{A} = \hat{i}_x - e^x\,\hat{i}_y - \cos ax\,\hat{i}_z$$

At the origin (0, 0, 0)

$$\nabla\times\vec{A} = \hat{i}_x - \hat{i}_y - \hat{i}_z$$

Problem 13

If ϕ is a scalar field and $\vec{A}$ is a vector field, prove that

$$\vec{\nabla}\times(\phi\vec{A}) = \phi(\nabla\times\vec{A}) + (\vec{\nabla}\phi)\times\vec{A}.$$

Solution :

$$\phi\vec{A} = \phi(A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k})$$

$$= \phi A_x\,\hat{i} + \phi A_y\,\hat{j} + \phi A_z\,\hat{k}$$

$$\vec{\nabla}\times(\phi\vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_x & \phi A_y & \phi A_z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(\phi A_z) - \frac{\partial}{\partial z}(\phi A_y)\right] + \hat{j}\left[\frac{\partial}{\partial z}(\phi A_x) - \frac{\partial}{\partial x}(\phi A_z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(\phi A_y) - \frac{\partial}{\partial y}(\phi A_x)\right]$$

$$= \hat{i}\left[\frac{\partial\phi}{\partial y}A_z + \phi\frac{\partial A_z}{\partial y} - \frac{\partial\phi}{\partial z}A_y - \phi\frac{\partial A_y}{\partial z}\right] + \hat{j}\left[\frac{\partial\phi}{\partial z}A_x + \frac{\partial A_x}{\partial z} - \frac{\partial\phi}{\partial x}A_z - \phi\frac{\partial A_z}{\partial x}\right] + \hat{k}\left[\frac{\partial\phi}{\partial x}A_y + \phi\frac{\partial A_y}{\partial x} - \frac{\partial\phi}{\partial y}A_x - \phi\frac{\partial A_y}{\partial y}\right]$$

$$= \phi\left[\hat{i}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\hat{j}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\hat{k}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_y}{\partial y}\right)\right]$$

$$+\hat{i}\left[A_z\frac{\partial \phi}{\partial y}-A_y\frac{\partial \phi}{\partial z}+\hat{j}\left(A_x\frac{\partial \phi}{\partial z}-A_z\frac{\partial \phi}{\partial x}\right)+\hat{k}\left(A_y\frac{\partial \phi}{\partial x}-A_x\frac{\partial \phi}{\partial y}\right)\right]$$

Now, $\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\hat{j}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\hat{k}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

and $(\vec{\nabla}\phi)\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$$= \hat{i}\left(A_z\frac{\partial \phi}{\partial y}-A_y\frac{\partial \phi}{\partial z}\right)+\hat{j}\left(A_x\frac{\partial \phi}{\partial z}-A_z\frac{\partial \phi}{\partial x}\right)+\hat{k}\left(A_y\frac{\partial \phi}{\partial x}-A_x\frac{\partial \phi}{\partial y}\right)$$

Hence,

$$\vec{\nabla}\times(\phi\vec{A}) = \phi(\nabla\times\vec{A})+(\vec{\nabla}\phi)\times\vec{A}. \qquad \text{Proved}$$

Problem 14

Show that $\vec{F} = (x+i)\hat{i}+(x+z)\hat{j}+(y-z)\hat{k}$ represents a conservative field.

Solution :

For a conservative field, $\vec{\nabla}\times\vec{F} = 0$.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y) & (x+z) & (y-z) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(y-z)-\frac{\partial}{\partial z}(x+z)\right]+\hat{j}\left[\frac{\partial}{\partial z}(x+y)-\frac{\partial}{\partial x}(y-z)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(x+z)-\frac{\partial}{\partial y}(x+y)\right]$$

$$= \hat{i}(1-1)+\hat{j}(0)+\hat{k}(1-1)$$

$$= 0$$

Hence, $\vec{F}$ is a conservative field.

Problem 15

Find the divergence and curl of a vector $\vec{A} = x^2y\hat{i}+(x-y)\hat{k}$.

Solution :

We have $\vec{A} = x^2y\hat{i}+(x-y)\hat{k}$.

$$\text{Div } \vec{A} = \vec{\nabla}\cdot\vec{A}$$

$$= \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot[x^2y\hat{i}+(x-y)\hat{k}]$$

$$= \frac{\partial}{\partial x}(x^2y)+\frac{\partial}{\partial z}(x-y) = 2xy$$

$$\text{Curl } \vec{A} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 0 & x-y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x-y)\right]+\hat{j}\left[\frac{\partial}{\partial z}(x^2y)-\frac{\partial}{\partial x}(x-y)\right]+\hat{k}\left[-\frac{\partial}{\partial y}(x^2y)\right]$$

$$= \hat{i}(-1)+\hat{j}(-1)+\hat{k}(x^2) = -\hat{i}-\hat{j}-x^2\hat{k}$$

Ans. : $\text{Div } \vec{A} = 2xy$

$\text{Curl } \vec{A} = -\hat{i}-\hat{j}-x^2\hat{k}$

Problem 16

Prove that the divergence of the electric field and that of electric flux density in a charge free region is zero.

Solution :

Data : In a charge free region, the charge density, $\rho = 0$.

Formula : $\vec{\nabla} \cdot \vec{D} = \rho$: Gauss' law of electrostatics

Calculations :

$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot \varepsilon_0 \vec{E} = \varepsilon_0 \vec{\nabla} \cdot \vec{E} = 0$$

Problem 17

Given that $\vec{D} = 10x \hat{i}_x$ (C/m^2). Determine the flux crossing $1\ m^2$ area that is normal to the x-axis at x = 3 m.

Solution :

Data : $D_x = 10x$, $x = 3$ m

Formula : Using Gauss' law, Total flux = $\int \vec{D} \cdot \vec{ds}$

Calculations :

$$\int \vec{D} \cdot \vec{ds} = \int (\hat{i}_x D_x + \hat{i}_y D_y + \hat{i}_z D_z) \cdot (\hat{i}_x dS_x + \hat{i}_y dS_y + \hat{i}_z dS_z)$$

$$= \int (D_x dS_x + D_y dS_y + D_z dS_z)$$

$$= \int D_x dS_x = D_x \int dS_x = D_x \cdot (1)$$

$$= 10x \cdot 1 = 10x$$

At x = 3 cm,

$$\text{Flux} = 10 \times 3 = 30$$

Result : Flux = 30 C.

Problem 18

Let $\vec{D} = 2y^2z^2 \hat{i}_x + 3xy^2z^2 \hat{i}_y + 2xyz \hat{i}_z$ (pC/m^2) in free space. Find :

(i) the total electrical flux passing through the surface $x = 2$, $0 \le y \le 2$ and $0 \le z \le 2$ in a direction away from the origin.

(ii) the total charge contained in an elemental sphere of a radius 1 μm centered at P (2, 2, 2).

Solution :

(i) ABCD is the plane specified. Its direction is $\hat{i}_x$ and an elementary plane dS on it is given by

$$dS = dy\, dz\, \hat{i}_x$$

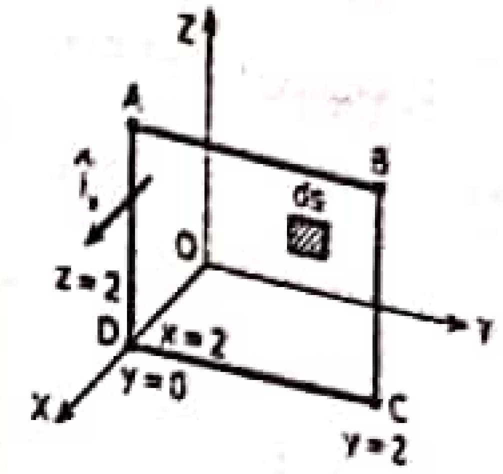


Figure for problem 18

The electrical flux is given by

$$\phi = \int_S \vec{D} \cdot \vec{ds}$$

For x = 2, $\vec{D} = 2y^2z^2 \hat{i}_x + 6y^2z^2 \hat{i}_y + 4yz \hat{i}_z$ (pC/m^2)

Hence, $$\phi = \int_{y=0}^{2} \int_{z=0}^{2} (2y^2z^2 \hat{i}_x + 6y^2z^2 \hat{i}_y + 4yz \hat{i}_z) \cdot dy\, dz$$

$$= \int_{y=0}^{2} \int_{z=0}^{2} 2y^2z^2\, dy\, dz$$

[Since, $\hat{i}_x \cdot \hat{i}_x = 1$, $\hat{i}_y \cdot \hat{i}_x = 0$, $\hat{i}_z \cdot \hat{i}_x = 0$]

$$= 2\int_0^2 y^2\, dy \int_0^2 z^2\, dz = 2\left(\frac{y^3}{3}\right)\Bigg|_0^2 \left(\frac{z^3}{3}\right)\Bigg|_0^2$$

$$\therefore \quad \phi = 2\left(\frac{8}{3}\right)\left(\frac{8}{3}\right) = 14.22$$

(ii) According to Gauss' law the volume charge, density, ρ_v is given by

$$\rho_v = \vec{\nabla} \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(2y^2z^2) + \frac{\partial}{\partial y}(3xy^2z^2) + \frac{\partial}{\partial z}(2xyz)$$

$$\therefore \quad \rho_v = 0 + 6xyz^2 + 2xy$$

At P (2, 2, 2), total charge is

$$\rho_v = 104 \ (pC/m^3)$$

The radius of the elemental sphere in 1 μm.

The volume, $\Delta V = \frac{4}{3}\pi(10^{-6})^3 \text{ m}^3$

The total charge contained in it is

$$Q = \rho_v \Delta V = 104 \cdot \frac{4}{3} \cdot \pi (10^{-6})^3$$

$$\therefore \quad Q = 4.36 \times 10^{-28} \text{ C.}$$

Problem 19

If $\vec{D} = 10x\hat{i} - 4y\hat{j} + Cz\hat{k}$, where C is a constant, find the value of C using Gauss law for a charge free region.

Solution :

Gauss' law : $\vec{\nabla} \cdot \vec{D} = \rho_v$ where ρ_v = volume charge density.

In a charge free region $\rho_v = 0$. Hence,

$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\therefore \quad \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (10x\hat{i} - 4y\hat{j} + Cz\hat{k}) = 0$$

$$\frac{\partial}{\partial x}(10x) + \frac{\partial}{\partial y}(-4y) + \frac{\partial}{\partial z}(Cz) = 0$$

$$10 - 4 + C = 0$$

Ans. : $C = -6$

Problem 20

If the magnetic field $\vec{H} = (3x\cos\beta + 6y\sin\alpha)\hat{k}$, find the current density $\vec{J}$ for steady fields.

Solution :

Ampere's circuital law is $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

For steady fields $\frac{\partial \vec{D}}{\partial t} = 0$.

$$\therefore \quad \vec{J} = \vec{\nabla} \times \vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & (3x\cos\beta + 6y\sin\alpha) \end{vmatrix}$$

$$= \hat{i}\frac{\partial}{\partial y}(3x\cos\beta + 6y\sin\alpha) - \hat{j}\frac{\partial}{\partial x}(3x\cos\beta + 6y\sin\alpha)$$

$$= 6\sin\alpha\,\hat{i} - 3\cos\beta\,\hat{j}$$

Problem 21

Find the volume charge density, ρ_v at P (1, 2, 3) in free space if the potential is given by $V = 50x^2yz + 20y^2$.

Solution :

By Gauss' law : $\vec{\nabla} \cdot \vec{D} = \rho_v$, $\vec{D} = \epsilon_0 \vec{E}$, $\vec{E} = -\vec{\nabla} V$

$$\vec{E} = -\vec{\nabla} V = -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (50x^2y + 20y^2)$$

$$= -[100xyz\,\hat{i} + (50x^2y + 40y)\hat{j} + 50x^2y\hat{k}]$$

$$\vec{D} = -\epsilon_0[100xyz\,\hat{i} + (50x^2y + 40y)\hat{j} + 50x^2y\hat{k}]$$

$$\rho_v = \vec{\nabla} \cdot \vec{D}$$

$$= -\epsilon_0\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot [100xyz\,\hat{i} + (50x^2y + 40y)\hat{j} + 50x^2y\hat{k}]$$

$$= -\epsilon_0[100yz + 40]$$

At P (1, 2, 3), $\rho_v = -640\,\epsilon_0$.

Problem 22

Given that $\vec{H} = H_m e^{j(\omega t + \beta z)}\hat{i}$ (A/m) in free space. Find $\vec{E}$.

Solution :

For free space, $\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

Now, $\vec{\nabla} \times \vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$

Given : $H_x = H_m e^{j(\omega t + \beta z)}$, $H_y = 0$, $H_z = 0$.

$$\therefore \quad \vec{\nabla} \times \vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix}$$

$$= \hat{i}[0-0] + \hat{j}\left[\frac{\partial H_x}{\partial z} - 0\right] + \hat{k}\left[0 - \frac{\partial H_x}{\partial y}\right]$$

$$= \hat{j}\frac{\partial}{\partial z}(H_m e^{j(\omega t+\beta z)}) - \hat{k}\frac{\partial}{\partial y}(H_m e^{j(\omega t+\beta z)})$$

$$\therefore \quad \vec{\nabla} \times \vec{H} = j\beta H_m e^{j(\omega t+\beta z)}\hat{j}$$

$$\therefore \quad \frac{\partial \vec{D}}{\partial t} = j\beta H_m e^{j(\omega t+\beta z)}\hat{j}$$

$$\vec{D} = j\beta H_m \int e^{j(\omega t+\beta z)} dt \cdot \hat{j}$$

$$= \frac{j\beta H_m}{\omega} \cdot e^{j(\omega t+\beta z)}\hat{j}$$

$$\therefore \quad \vec{D} = \frac{j\beta}{\omega} \cdot \vec{H} = \frac{j\beta}{\omega} H_m e^{j(\omega t+\beta z)}\hat{j}$$

Again $\vec{D} = \epsilon_0 \vec{E}$ in free space.

So, $\vec{E} = \frac{j\beta H_m}{\omega \epsilon_0} \cdot e^{j(\omega t+\beta z)}\hat{j}$

Problem 23

Find the divergence and curl of a vector field $\vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$.

Solution :

$$\text{Div } \vec{A} = \vec{\nabla} \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (x^2\hat{i} + y^2\hat{j} + z^2\hat{k})$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$\text{Curl } \vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(z^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= 0$$

Problem 24

A region is specified by the potential function $\phi = 4x^2 + 3y^2 - 9z^2$. Calculate the electric field strength at any point (3, 4, 5) in this region.

Solution :

$$\vec{E} = -\vec{\nabla}\phi = -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$= -\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

Let $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$

Here, $E_x = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}(4x^2 + 3y^2 - 9z^2) = 8x$

$E_y = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}(4x^2 + 3y^2 - 9z^2) = 6y$

$E_z = -\frac{\partial \phi}{\partial z} = -\frac{\partial}{\partial z}(4x^2 + 3y^2 - 9z^2) = -18z$

Now, $$\vec{\nabla} \times \vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

Given : $H_x = H_m e^{j(\omega t + \beta z)}$, $H_y = 0$, $H_z = 0$.

$$\therefore \quad \vec{\nabla} \times \vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix}$$

$$= \hat{i}[0-0] + \hat{j}\left[\frac{\partial H_x}{\partial z} - 0\right] + \hat{k}\left[0 - \frac{\partial H_x}{\partial y}\right]$$

$$= \hat{j}\frac{\partial}{\partial z}(H_m e^{j(\omega t+\beta z)}) - \hat{k}\frac{\partial}{\partial y}(H_m e^{j(\omega t+\beta z)})$$

$$\therefore \quad \vec{\nabla} \times \vec{H} = j\beta H_m e^{j(\omega t+\beta z)}\,\hat{j}$$

$$\therefore \quad \frac{\partial \vec{D}}{\partial t} = j\beta H_m e^{j(\omega t+\beta z)}\,\hat{j}$$

$$\vec{D} = j\beta H_m \int e^{j(\omega t+\beta z)} dt \cdot \hat{j}$$

$$= \frac{j\beta H_m}{\omega} \cdot e^{j(\omega t+\beta z)}\,\hat{j}$$

$$\therefore \quad \vec{D} = \frac{j\beta}{\omega} \cdot \vec{H} = \frac{j\beta}{\omega} H_m e^{j(\omega t+\beta z)}\,\hat{j}$$

Again $\vec{D} = \epsilon_0 \vec{E}$ in free space.

So, $$\vec{E} = \frac{j\beta H_m}{\omega \epsilon_0} \cdot e^{j(\omega t+\beta z)}\,\hat{j}$$

Problem 23

Find the divergence and curl of a vector field $\vec{A} = x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}$.

Solution :

$$\text{Div } \vec{A} = \vec{\nabla} \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (x^2\hat{i} + y^2\hat{j} + z^2\hat{k})$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$\text{Curl } \vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(z^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= 0$$

Problem 24

A region is specified by the potential function $\phi = 4x^2 + 3y^2 - 9z^2$. Calculate the electric field strength at any point (3, 4, 5) in this region.

Solution :

$$\vec{E} = -\vec{\nabla}\phi = -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$= -\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

Let $\vec{E} = E_x\,\hat{i} + E_y\,\hat{j} + E_z\,\hat{k}$

Here, $$E_x = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}(4x^2 + 3y^2 - 9z^2) = 8x$$

$$E_y = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}(4x^2 + 3y^2 - 9z^2) = 6y$$

$$E_z = -\frac{\partial \phi}{\partial z} = -\frac{\partial}{\partial z}(4x^2 + 3y^2 - 9z^2) = -18z$$

$$\therefore \quad \vec{E} = 8x\,\hat{i} + 6y\,\hat{j} - 18z\,\hat{k}$$

At point (3, 4, 5),

$$\vec{E} = 24\,\hat{i} + 24\,\hat{j} - 90\,\hat{k}$$

Problem 25

Determine whether or not the following pair of electric and magnetic fields satisfy Maxwell's equation in free space.

$$\vec{E} = 2y\,\hat{j}, \qquad \vec{H} = 5x\,\hat{i}$$

Solution :

First equation : $\vec{\nabla} \cdot \vec{D} = 0$

$$\vec{\nabla} \cdot \vec{D} = \epsilon_0\, \vec{\nabla} \cdot \vec{E} = \epsilon_0 \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot 2y\,\hat{j}$$

$$= \epsilon_0 \frac{\partial}{\partial y}(2y) = 2\epsilon_0 \neq 0$$

Hence, not satisfied.

Second equation : $\vec{\nabla} \cdot \vec{B} = 0$

$$\vec{\nabla} \cdot \vec{B} = \mu_0\, \vec{\nabla} \cdot \vec{H} = \mu_0 \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot (5x\,\hat{i})$$

$$= \mu_0 \frac{\partial}{\partial x}(5x) = 5\mu_0 \neq 0$$

Hence, not satisfied.

Third equation : $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\text{L.H.S.} = \vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2y & 0 \end{vmatrix}$$

$$= \hat{i}\left[0 - \frac{\partial}{\partial z}(2y)\right] + \hat{j}[0-0] + \hat{k}\left[\frac{\partial}{\partial x}(2y) - 0\right]$$

$\therefore$ L.H.S. $= 0$

Now, R.H.S. $= -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t} = -\mu_0 \frac{\partial}{\partial t}(5x\,\hat{i}) = 0$

$\therefore$ L.H.S. = R.H.S.

Hence, third equation is satisfied.

Fourth equation : $\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

$$\text{L.H.S.} = \vec{\nabla} \times \vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5x & 0 & 0 \end{vmatrix}$$

$$= \hat{i}[0-0] + + \hat{j}\left[\frac{\partial}{\partial z}(5x) - 0\right] + \hat{k}\left[0 - \frac{\partial}{\partial y}(5x)\right]$$

$\therefore$ L.H.S. $= 0$

Now, R.H.S. $= \frac{\partial \vec{D}}{\partial t} = \epsilon_0 \frac{\partial}{\partial t}\vec{D} = \epsilon_0 \frac{\partial}{\partial t}(2y\,\hat{i}) = 0$

$\therefore$ L.H.S. = R.H.S.

Hence, fourth equation is satisfied.

Problem 26

Given $\vec{E} = E_m \sin(\omega t - \beta z)\,\hat{i}_y$ in free space. Find $\vec{D}$, $\vec{B}$ and $\vec{H}$.

Solution :

In free space, $\varepsilon_r = 1$, $\mu_r = 1$.

$$\vec{D} = \varepsilon_0\, \vec{E} = \varepsilon_0\, E_m \sin(\omega t - \beta z)\,\hat{i}_y \quad (\text{C/m}^2)$$

According to Maxwell's equation, for Faraday's law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Hence, $$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$$

$$\frac{\partial \vec{B}}{\partial t} = \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

Given $\vec{E} = E_y \hat{i}_y$, with $E_x = 0, E_z = 0, E_y = E_m \sin(\omega t - \beta z)$.

$$\frac{\partial \vec{B}}{\partial t} = \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = \frac{\partial E_y}{\partial z}\hat{i}_x + \frac{\partial E_y}{\partial x}\hat{i}_z$$

Here, $$\frac{\partial E_y}{\partial z} = -E_m \beta \cdot \cos(\omega t - \beta z).$$

$$\frac{\partial E_y}{\partial x} = 0$$

Hence, $$\frac{\partial \vec{B}}{\partial t} = -E_m \beta \cos(\omega t - \beta z)\hat{i}_x$$

$$\partial \vec{B} = -E_m \beta \cos(\omega t - \beta z)\hat{i}_x \cdot \partial t$$

$$\vec{B} = -\int E_m \beta \cos(\omega t - \beta z)\, dt \cdot \hat{i}_x$$

$$\vec{B} = -\frac{E_m \beta}{\omega}\sin(\omega t - \beta z)\hat{i}_x \quad Wb/m^2$$

Again, $\vec{B} = \mu_o \vec{H}$ in free space.

Hence, $$\vec{H} = -\frac{\beta E_m}{\omega \mu_o}\sin(\omega t - \beta z)\hat{i}_x \quad (A/m)$$

Important Points to Remember

1. Parameters of Cartesian Coordinates :

Position vector : $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

Length element : $d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Surface elements : $d\vec{s}_x = \pm dy\,dz\,\hat{i}$, $d\vec{s}_y = dz\,dx\,\hat{j}$, $d\vec{s}_z = dx\,dy\,\hat{k}$

Volume element : $dV = dx\,dy\,dz$

Gradient : $\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}$

Divergence : $\vec{\nabla} \cdot \vec{B} = \frac{\partial B_x}{\partial x}\hat{i} + \frac{\partial B_y}{\partial y}\hat{j} + \frac{\partial B_z}{\partial z}\hat{k}$

Curl : $\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$

2. Divergence theorem : $\int_V \vec{\nabla} \cdot \vec{A}\, dV = \oint_S \vec{A} \cdot d\vec{s}$

3. Stoke's theorem : $\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$

4. Basic postulates of electrostatics :

$\nabla \cdot \vec{D} = \rho$,	$\oint_S \vec{D} \cdot d\vec{s} = Q$	: Gauss' law of electrostatics
$\nabla \cdot \vec{B} = 0$,	$\int \vec{B} \cdot d\vec{s} = 0$	: Gauss' law of magnetostatic
$\vec{\nabla} \times \vec{E} = 0$,	$\oint_C \vec{E} \cdot d\vec{l} = 0$	: Faraday's law
$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$,	$\oint \vec{B} \cdot d\vec{l} = \mu_o I$	: Ampere's circuital law

5. Maxwell's equations

Differential (Point) form	Integral form	Significance
$\vec{\nabla} \cdot \vec{D} = \rho$	$\oint \vec{D} \cdot \vec{ds} = \int_V \rho \, dv$	Gauss's law for electrostatics
$\vec{\nabla} \cdot \vec{B} = 0$	$\oint \vec{B} \cdot \vec{ds} = 0$	Gauss's law for magnetostatics (non-existance of magnetic monopole)
$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}$	$\oint \vec{E} \cdot \vec{dl} = -\int_S \dot{\vec{B}} \cdot \vec{ds}$	Faraday's law
$\vec{\nabla} \times \vec{H} = \vec{J} + \dot{\vec{D}}$	$\oint \vec{H} \cdot \vec{dl} = \int_S (\vec{J} + \dot{\vec{D}}) \cdot \vec{ds}$	Ampere's law
Supplementary equation		
$\vec{\nabla} \cdot \vec{J} = -\dot{\rho}$	$\oint_S \vec{J} \cdot \vec{ds} = -\int_V \rho \, dv$	Continuity equation

6. Maxwell's equations for free space

Differential (Point) form	Integral form	Significance
$\vec{\nabla} \cdot \vec{D} = 0$	$\oint \vec{D} \cdot \vec{ds} = 0$	Gauss's law for electrostatics
$\vec{\nabla} \cdot \vec{B} = 0$	$\oint \vec{B} \cdot \vec{ds} = 0$	Gauss's law for magnetostatics (non-existance of magnetic monopole)
$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}$	$\oint \vec{E} \cdot \vec{dl} = -\int_S \dot{\vec{B}} \cdot \vec{ds}$	Faraday's law
$\vec{\nabla} \times \vec{H} = \vec{J} + \dot{\vec{D}}$	$\oint \vec{H} \cdot \vec{dl} = \int_S \dot{\vec{D}} \cdot \vec{ds}$	Ampere's law

Review Questions

1. Explain Cartesian coordinate system. [Refer § 4.2.1]
2. Write down the surface vectors for (a) Cartesian, (b) Cylindrical, (c) Spherical coordinate systems. [Refer § 4.2.1 (C), 4.2.2 (C), 4.2.3 (C)]
3. Define a field. What are scalar and vector fields? [Refer § 4.3]
4. What is gradient of a scalar field? Present it in Cartesian coordinates. [Refer § 4.4.2]
5. Define and explain divergence of a vector field. Express it in Cartesian coordinates. [Refer § 4.4.3]
6. What is the significance of the curl of a vector field. Explain it with an example. Express it in Cartesian coordinates. [Refer § 4.4.3]
7. Show that the divergence of the curl of a vector is zero. [Refer § 4.4.5]
8. Define static and time varying fields. [Refer § 4.6.1]
9. Derive Gauss' law for static electric field in differential and in integral form. [Refer § 4.6.2 (A)]
10. State Gauss' law for static magnetic field in differential and in integral form. [Refer § 4.6.2 (B)]
11. State the differential and integral form of Faraday's law for static electric field. [Refer § 4.6.2 (C)]
12. Obtain Ampere's circuital law for static magnetic field in differential and integral form. [Refer § 4.6.2 (D)]
13. Write down the continuity equation and state its significance. [Refer § 4.6.2 (E)]
14. Obtain Faraday's law for time varying fields in differential and integral form. [Refer § 4.6.4 (A)]
15. Obtain Ampere's law for time varying fields in differential and integral form. [Refer § 4.6.4 (B)]
16. Derive point form of all Maxwell's equations and state their significance. [Refer § 4.6]
17. Derive integral form of all Maxwell's equations and state their significance. [Refer § 4.6]

EXERCISE

1. Two points A (2, 2, 1) and B (3, – 4, 2) are given in the Cartesian system. Calculate the vector from A to B and a unit vector directed from A to B.

[Ans. : $\vec{AB} = \hat{i}_x - 6\hat{i}_y + \hat{i}_z$, $\hat{i}_{AB} = 0.1622\,\hat{i}_x - 0.9733\,\hat{i}_y + 0.1622\,\hat{i}_z$]

2. Using Gauss' law, find the total charge in a volume enclosed by the six planes defined as $1 \le x \le 2$, $2 \le y \le 3$, $3 \le z \le 4$ if

$\vec{D} = 4x\,\hat{i}_x + 3y^2\,\hat{i}_y + 2z^3\,\hat{i}_z$ C / m^2. [Ans. : 93 C]

3. Determine the net flux of the vector field $\vec{D}(x, y, z) = 2x^2 y\,\hat{i}_x + 2\,\hat{i}_y + y\,\hat{i}_z$ emerging from the unit cube defined by $0 \le x, y, z \le 1$. [Ans. : 1 C]

4. Given that, $\vec{B} = 2.5\left(\sin\frac{\pi x}{2}\right)e^{-2y}\,\hat{i}_z$ Wb / m^2. Find the total magnetic flux crossing the strip defined by $0 \le x \le 2$ m, $y \ge 0$, $z = 0$. [Ans. : 1.5915 Wb]

5. Do the fields $\vec{E} = E_m \sin x \sin t\,\hat{i}_y$ and $\vec{H} = \frac{E_m}{\mu_o}\cos x \cos t\,\hat{i}_z$ satisfy Maxwell's equations? [Ans. : Yes]

6. Given, $\vec{H} = H_m e^{j(\omega t + \beta z)}\cdot\hat{i}_x$ A / m in free space. Find $\vec{E}$.

[Ans. : $\vec{E} = \frac{\beta H_m}{\varepsilon_o \omega} e^{j(\omega t + \beta z)}\,\hat{i}_y$ V / m]

7. Given, $\vec{E} = E_m \sin(\omega t - \beta z)\,\hat{i}_y$ in free space. Find $\vec{D}$, $\vec{B}$ and $\vec{H}$.

[Ans. : $\vec{D} = \varepsilon_o E_m \sin(\omega t - \beta z)\,\hat{i}_y$ C / m^2

$\vec{B} = -\frac{\beta}{\omega} E_m \sin(\omega t - \beta z)\,\hat{i}_x$ Wb / m^2

$\vec{H} = -\frac{\beta}{\omega \mu_o} E_m \sin(\omega t - \beta z)\,\hat{i}_x$ Wb / m^2]

Previous University Examination Questions with Solutions

1. What is divergence of a vector field? Express it in Cartesian Coordinate System.
[Refer § 3.4.3] (M.U. May 2017) (3 m)

2. Show that the divergence of a curl of a vector is zero. (M.U. Dec. 2017, 19) (3 m)
[Refer § 3.4.5]

3. Explain the physical significance of divergence and curl of a vector field.
[Refer § 3.4.3 and 3.4.4] (M.U. Dec. 2019) (5 m)

4. Derive Maxwell's two general equations in integral and differential form.
[Refer § 3.5] (M.U. Dec. 2017) (5 m)

5. Derive Maxwell's third equation. (M.U. Nov. 2018) (5 m)
[Refer § 3.5.2 (c), 3.5.4 (A)]

6. Write Maxwell's equations in differential form and give their physical significance.
[Refer § 3.5.5] (M.U. May 2019) (5 m)

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MODULE 4

Relativity

(Prerequisites : Cartesian co-ordinate system)

Special theory of Relativity : Inertial and Non-inertial Frames of reference, Galilean transformations, Lorentz transformations (Space-time coordinates), Time Dilation, Length Contraction and Mass-Energy relation. (02 Hours)

(Weightage - 10%)

Course Outcome : CO4 : Learner will be able to explain the fundamentals of relativity.

SYNOPSIS

4.1 Introduction

4.2 Einsten's Classical Theory of Relativity (Newtonian Theory of Relativity)

4.3 Einstein's Special Theory of Relativity

4.4 Time Dilation

4.5 Length Contraction

4.6 Einstein's Mass-Energy Relation

4.7 Important Points to Remembers

4.8 Problems

4.9 Review Questions

4.1 Introduction

What is relativity?

Consider a train moving with a speed of 60 km/hour. The train is observed by three observers.

(i) The first observer is standing at the station.

(ii) The second observer is moving in the direction of the train with a velocity of 20 km/hour.

(iii) The third observer is moving with a velocity of 20 km/hour in the opposite direction of the train.

The observations of the three observers are different as follows :

(i) The first observer would observe the velocity of the train as 60 km/hour.

(ii) The second observer would observe the velocity of the train as (60 – 20 =) 40 km/hour.

(iii) The third observer would observe the velocity of the train as (60 + 20 =) 80 km/hour.

The oldest theory of Physics is the Classical Physics or Newtonian Physics that deals with the absolute motion of an object considering space and time to be absolute and two separate entities. However, this concept failed to explain the motion with high velocities very close to the velocity of light.

The development of theory of relativity by Einstein in 1905 revolutionalized the old concepts. It discards the concept of absolute motion and deals with objects and observers moving with high velocities (~ c) and relative velocities with respect to each other. This theory was developed in two steps and thus are divided into two parts.

(i) Einstein's Classical Theory of Relativity based on Classical Physics, *i.e.,* Newtonian mechanics.

(ii) Einstein's Special Theory of Relativity applicable to all laws of Physics.

4.2 Einsten's Classical Theory of Relativity (Newtonian Theory of Relativity)

Einstein initially developed his theory of relativity for classical physics, *i.e.* Newtonian Mechanics. This is called Einstein's classical theory of relativity.

4.2.1 : Frame of Reference

The motion of an object can be described only with the help of a coordinate system. The coordinate system in such cases is known as the frame of reference. There are two types of frame of reference.

(1) Inertial frame of reference or unaccelerated frame

A frame of reference is said to be inertial when objects in this frame obey Newton's law of inertia and other laws of Newtonian mechanics. In this frame an object is not acted upon by an external force. It is at rest or moves with a constant velocity.

(2) Non inertial frame

A frame of reference which is in an accelerated motion with respected to an inertial frame of reference is called a non-inertial frame of reference. In such frame an object even without an external force acting on it, is accelerated. In non-inertial frame the Newton's laws are not valid.

Example : A ball placed the floor of a train moves to the rear if the train accelerates forward even though no forces act on it. In this case, the train moves in an inertial frame of reference and the ball is in a non-inertial frame of reference.

4.2.2 : Galilean Transformations

The transformation from one inertial frame of reference to another is called Galilean transformation. Knowing the laws of motion of an object in a reference system S, the laws of motion of the same object in another reference system S' can be derived.

Let us consider a physical event. An event is something that happens without depending on the reference frame used to describe it. Suppose a collision of two particles occur at a point (x, y, z) at an instant of t secs. We describe this event by the coordinates (x, y, z, t) in one frame of reference, say, in a laboratory on the earth. The same event observed from a different reference frame, *e.g.,* from an aircraft flying overhead would also be specified by a set of four coordinates in space and time (x', y', z', t') which is different from the earlier set of (x, y, z, t).

Consider now two observers O and P, where P travels with a constant velocity 'v' with respect to O along their common X-X' axis. Here E is the event specified by coordinates (x, y, z, t) and (x', y', z', t') in frames S and S' respectively.

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Consider now two observers O and P, where P travels with a constant velocity 'v' with respect to O along their common X-X' axis. Here E is the event specified by coordinates (x, y, z, t) and (x', y', z', t') in frames S and S' respectively.

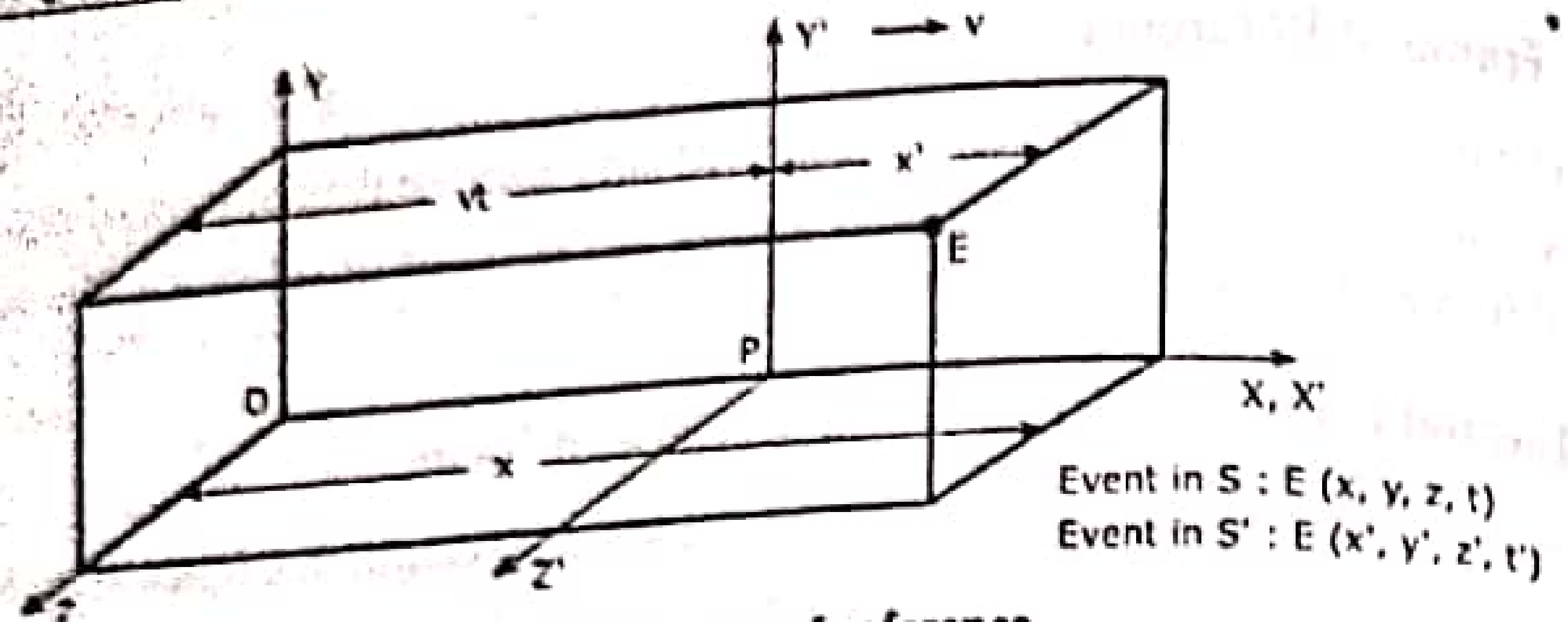


Fig. 4.1 : Frame of reference

(a) Galilean Coordinate Transformations

From Fig. 4.1, it is observed that

$$x' = x - vt, \; y' = y, \; z' = z \text{ and } t' = t \qquad \ldots (4.1)$$

These four equations are called Galilean coordinates transformations.

(b) Galilean Velocity Transformations

The velocity coordinates of the object in event E can be assigned as (u_x, u_y, u_z) and (u_x', u_y', u_z') in frame S and in frame S' respectively. Then from equation (4.1), it can be written as

$$u_x' = \frac{dx'}{dt'} = \frac{d}{dt}(x - vt)\frac{dt}{dt'} = \frac{dx}{dt} - v = u_x - v \qquad \text{as } \frac{dt}{dt'} = 1$$

Altogether, the Galilean velocity transformation are

$$u_x' = u_x - v, \quad u_y' = v_y, \quad u_z' = v_z \qquad \ldots (4.2)$$

(c) Galilean Acceleration Transformation

In inertial frames of reference S and S', the acceleration components remain the same. Thus,

$$a_x' = a_x, \quad a_y' = a_y, \quad a_z' = a_z \qquad \ldots (4.3)$$

4.3 Einstein's Special Theory of Relativity

Einstein observed that his Classical Theory of Relativity fails for very high speed ($v \sim c$) particles. This is due to the fact that in Newtonian mechanics, there is no limit, in principle, to the allowed speed of a particle. In 1905, he extended his Classical Theory of Relativity to include all the laws of Physics and Special Theory of Relativity was developed

The special theory of relativity deals with the problems in which one frame of reference moves with a constant linear velocity relative to another frame of reference.

4.3.1 : Postulates of Special Theory of Relativity

Einstein in his Special Theory of Relativity postulated that

(i) **All the fundamental laws of physics retain the same form in all the inertial frames of reference.**

(ii) **The velocity of light in free space is constant and is independent of the relative motion of the source and the observer in any frame of reference.**

4.3.2 : Einstein proved the following facts based on his theory of relativity

Let v be the velocity of a spaceship with respect to a given frame of reference where an observer makes his observations.

(a) All clocks on the spaceship will go slow by a factor $\sqrt{1-(v^2/c^2)}$.

(b) The mass of the spaceship increases by a factor $\left[1-(v^2/c^2)\right]^{-1/2}$.

(c) All objects on the spaceship will be contracted by a factor $\sqrt{1-(v^2/c^2)}$.

(d) The speed of a material object can never exceed the velocity of light.

(e) Mass and energy are interconvertible,

$$E = mc^2$$

(f) If two objects A and B are moving with velocities u and v respectively along the X-axis, the relative velocity of A with respect to B is given by

$$v_R = \frac{u - v}{1 - (uv/c^2)}$$

Here, u and v are both comparable with the value of c.

4.3.3 : Lorentz Transformation of Space and Time

In Newtonian mechanics, the Galilean transformations expressed in equations (4.1), (4.2) and (4.3) relate the space and time coordinates in one inertial frame to those the other frame. However, these equations are not valid for cases where the object velocity v approaches the value of c, the velocity of light. The transformation equations apply for all velocities upto c and incorporate the invariance of the speed of light were developed in 1890 by Lorentz. These are known as Lorentz transformations.

Let us consider two inertial frames S and S' as shown in the Fig. 4.2. The frame S' moves with a velocity v with respect to S in the positive X direction.

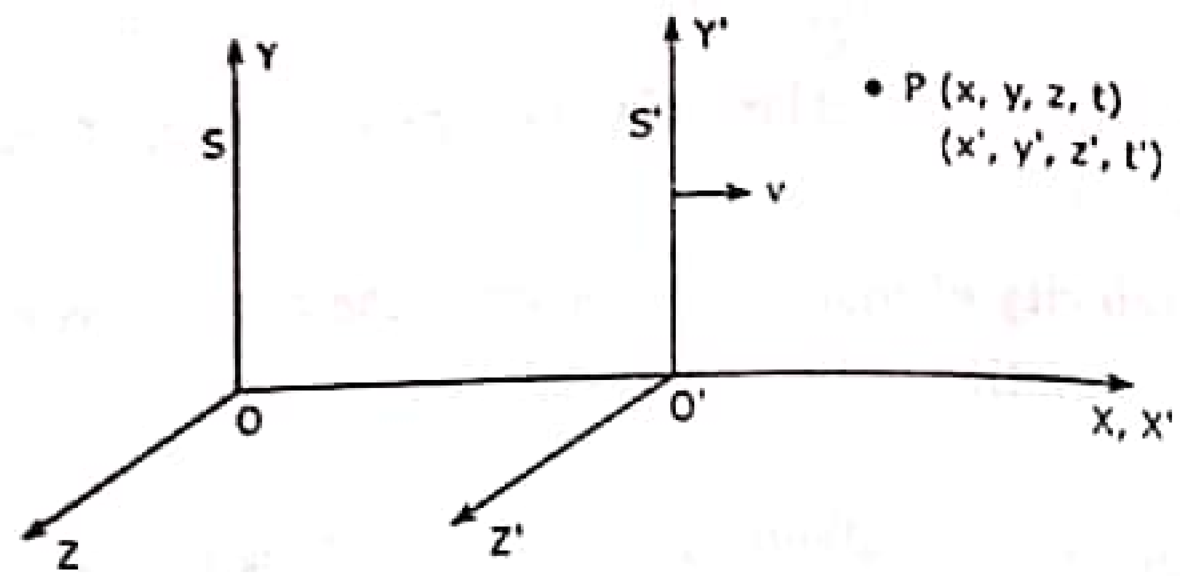


Fig. 4.2

Consider two observers O and O' situated at the origin in the frames S and S' respectively. Two coordinate systems coincide initially at the instant t = t' = 0. Suppose optical source is v kept at the common origin of the two frames. Let the source release a pulse at t = t' = 0 and at the same instant frame S' starts moving with a constant velocity v along +X direction, relative to frame S. This pulse reaches a point P with coordinates (x, y, z, t) and (x', y', z', t') in frames S and S' respectively.

Since S' is moving along +X direction with respect to S, the transformation equation of x and x' can be written as

$$x' = k(x - vt) \qquad \text{.......... (4.4)}$$

where, k is the constant of proportionality.

The inverse relation can be written as,

$$x = k(x' + vt') \qquad \text{.......... (4.5)}$$

Putting equation (4.4) in equation (4.5), we can write

$$x = k[k(x - vt) + vt']$$

$$\therefore \quad t' = kt - \frac{kx}{v}\left(1 - \frac{1}{k^2}\right) \qquad \text{.......... (4.6)}$$

Now, according to the second postulate of relativity, the speed of light c remains constant. So the velocity of the light pulse spreading out from the common origin observed by observers O and O' should be the same

$$x = ct$$

$$x' = ct' \qquad \text{.......... (4.7)}$$

Substituting equation (4.7) in equation (4.4) and equation (4.5), we have

$$ct' = k(c - v)t \qquad \text{.......... (4.8)}$$

and

$$ct = k(c + v)t' \qquad \text{.......... (4.9)}$$

Multiplying equations (4.8) with equation (4.9), we have

$$\therefore \quad k^2 = \frac{c^2}{c^2 - v^2}$$

$$\therefore \quad k = \pm\frac{1}{\sqrt{1 - (v^2/c^2)}} \qquad \text{.......... (4.10)}$$

and

$$1 - \frac{1}{k^2} = \frac{v^2}{c^2}$$

Using equations (4.10) in equation (4.4), we have

$$x' = \frac{x - vt}{\sqrt{1 - (v^2/c^2)}} \qquad \text{.......... (4.11)}$$

Substituting equations (4.10) and (4.11), we have

$$t' = \frac{t - (xv/c^2)}{\sqrt{1 - (v^2/c^2)}} \qquad \text{.......... (4.12)}$$

Hence, if the frame S' moves with a velocity v in +X direction with respect to the frame S, the transformation equations are,

$$x' = \frac{x - vt}{\sqrt{1 - (v^2/c^2)}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - (xv/c^2)}{\sqrt{1 - (v^2/c^2)}} \qquad \text{.......... (4.13)}$$

On the otherhand, if the frame S moves with a velocity v in – X direction with respect to the frame S', we get the inverse transformation equations as

$$x = \frac{x' + vt'}{\sqrt{1 - (v^2/c^2)}}, \quad y = y', \quad z = z', \quad t = \frac{t' - (x'v/c^2)}{\sqrt{1 - (v^2/c^2)}} \qquad \text{.......... (4.14)}$$

If the speed of the moving frame is much smaller than the velocity of light, i.e., if $v \ll c$, the Lorentz transformation equations reduce to Galilean transformation equations.

4.4 Time Dilation

The meaning of time dilation is extension of time. Time dilation is a difference in the elapsed time measured by two clocks due to a relative motion between them. To explain it let us consider two frames of reference S and S' with S' moving with a velocity v along X direction with respect to S as shown in Fig. 4.3. Imagine a gun placed at a fixed position P (x', y', z') in the frame S'. Suppose it fires two shots at instants t_1' and t_2' measured by the observer O' in the frame S'.

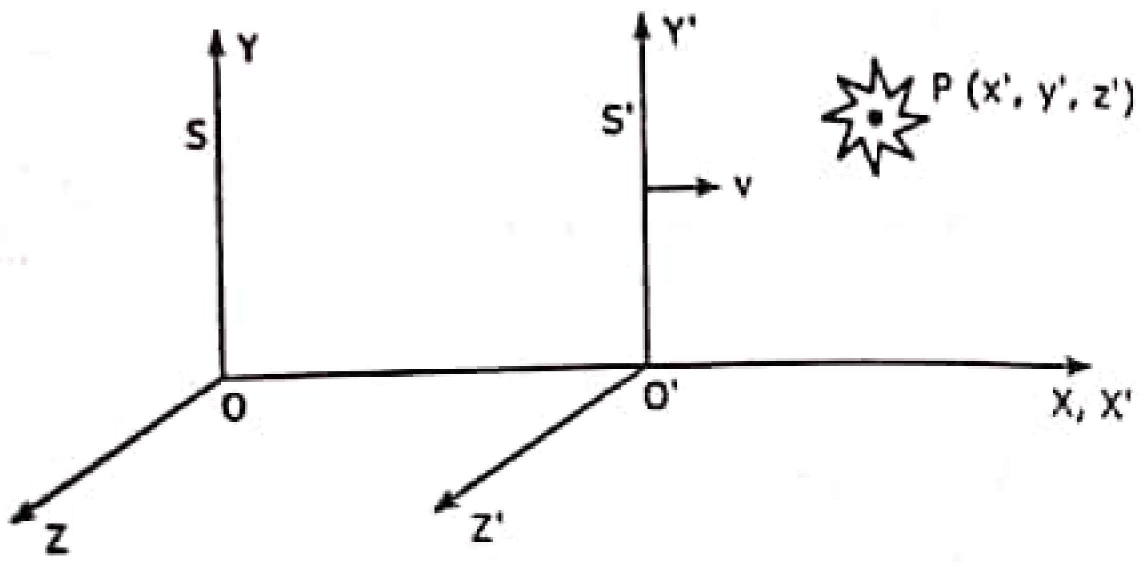


Fig. 4.3 : Time dilation

The time interval $(t_2' - t_1')$ of the two shots measured by O' at rest in the moving frame S' is called the proper time interval and is given by

$$T_0 = t_2' - t_1' \qquad \text{.....(4.15)}$$

As the motion between the two frames is relative, we may assume that the frame S is moving with velocity – v along the – X direction relative to frame S'. In frame S, the observer O who is at rest hears these two shots at different times t_1 and t_2.

The time interval appears to him is given by

$$t = t_2 - t_1 \qquad \text{.....(4.16)}$$

From inverse Lorentz transformation equations, we get

$$t_1 = \frac{t_1' + (vx'/c^2)}{\sqrt{1-(v^2/c^2)}} \qquad \text{.....(4.17)}$$

$$t_2 = \frac{t_2' + (vx'/c^2)}{\sqrt{1-(v^2/c^2)}} \qquad \text{.....(4.18)}$$

Substituting equations (4.17) and (4.18) in equation (4.16), we get

$$T = \frac{t_2' - t_1'}{\sqrt{1-(v^2/c^2)}} \qquad \text{.....(4.19)}$$

Using equation (4.15) in equation (4.19), we have

$$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}} \qquad \text{.....(4.20)}$$

which shows that $T > T_o$.

Here, T_o is called the **proper time which is defined as the time measured in the frame of reference in which the object is at rest.**

This verifies that the actual time interval in the moving frame appears to be lengthened by a factor $\frac{1}{\sqrt{v^2 - c^2}}$ when it is measured by an observer in the fixed frame, v being the relative velocity between the two frames.

4.5 Length Contraction

In classical mechanics the length of an object is independent of the velocity of the observer moving relative to the object. However, in the theory of relativity, the length of an object depends on the relative velocity between the observer and the object.

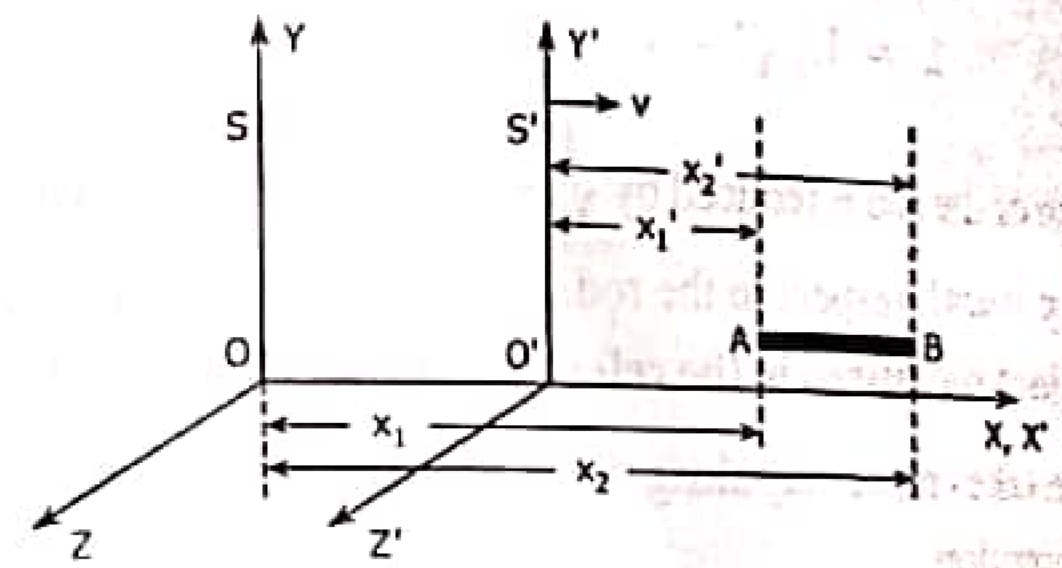


Fig. 4.4

To explain this, let us consider two inertial frames S and S' with S' moving with a velocity v in the X direction with respect to S.

Let a rod AB be at rest the moving frame S'. Its actual length is L_o at any instant as measured by the observer O' also at rest in the frame S'. So,

$$L_o = x_2' - x_1' \qquad \text{.....(4.21)}$$

where, x_1' and x_2' are the x coordinates of the rod in frame S' as shown in the Fig. 4.4.

At the same time, the length of AB measured by an observer O in the stationary frame S is given by

$$L = x_2 - x_1 \quad \text{......(4.22)}$$

x_1 and x_2 being the x coordinates of the rod in frame S.

From Lorentz transformation,

$$x_1' = \frac{x_1 - vt}{\sqrt{1-(v^2/c^2)}} \quad \text{......(4.23)}$$

$$x_2' = \frac{x_2 - vt}{\sqrt{1-(v^2/c^2)}} \quad \text{......(4.24)}$$

Substituting equations (4.23) and (4.24) in equation (4.21), we get the actual length as

$$L_o = \frac{x_2 - x_1}{\sqrt{1-(v^2/c^2)}} \quad \text{......(4.25)}$$

Using equation (4.22) in equation (4.25), we have

$$L_o = \frac{L}{\sqrt{1-(v^2/c^2)}}$$

$$\therefore \quad L = L_o\sqrt{1-(v^2/c^2)} \quad \text{......(4.26)}$$

Thus, the length of the rod is reduced by $\sqrt{1-(v^2/c^2)}$ when measured by an observer moving with velocity v with respect to the rod. Here, L_o is the proper length defined as **the length of the object measured in the reference frame in which the object is at rest.**

The contraction takes place only along the direction of motion and remains unchanged in a perpendicular direction.

4.6 Einstein's Mass-Energy Relation

In classical mechanics, the mass of a particle is independent of its velocity but in Einstein's special theory of relativity, the mass of a moving object depends upon its velocity and is given by

$$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$$

where, m_o is the rest mass and v is the velocity of the moving body and c is the velocity of light.

The increase in the energy of a particle by the applications of force may be estimated by the work done on it.

If a particle is displaced by a distance dx on the application of a force F, the kinetic energy dE generated and stored in it is given by the work done,

$$dE = dW = F\,dx \quad \text{......(4.27)}$$

Now, the force is defined as the time rate of change of momentum of the particle, by Newton's second law. Hence,

$$F = \frac{d(mv)}{dt} \quad \text{......(4.28)}$$

where, m is the mass of the particle and v is its velocity with which it moves on the application of the force F.

Thus, combining equations (4.27) and (4.28), we get

$$dE = \frac{d(mv)}{dt}\cdot dx$$

$$dE = \frac{dx}{dt}\,d(mv) = v\,[m\,dv + v\,dm]$$

$$\therefore \quad dE = mv\,dv + v^2\,dm \quad \text{......(4.29)}$$

Again,

$$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$$

So,

$$m^2 = \frac{m_o^2}{1-(v^2/c^2)}$$

$$m^2c^2 - m^2v^2 = m_o^2c^2 \quad \text{......(4.30)}$$

Differentiating equation (4.30), with m_o and c constants, we have

$$2m\,dm\,c^2 - 2m\,dm\,v^2 - 2v\,dv\,m^2 = 0$$

$$\therefore \quad dm\,c^2 = v^2\,dm + mv\,dv \quad \text{......(4.31)}$$

Substituting equation (4.31) in equation (4.29), we get

$$dE = dm\,c^2 \quad \text{......(4.32)}$$

Showing that the change in kinetic energy is directly proportional to the change in mass of the particle.

From equation (4.30), it is obvious that for a rest object v = 0 and mass $m = m_0$, the rest mass.

If the particle moves with a velocity v, its mass become m and its kinetic energy becomes E_k. Therefore, integrating equation (4.32), we get

$$\int_0^{E_k} dE = c^2 \int_{m_0}^{m} dm$$

$$E_k = c^2 (m - m_0)$$

$$\therefore \quad E_k = mc^2 - m_0 c^2 \quad \text{.......... (4.33)}$$

$$\therefore \quad mc^2 = E_k + m_0 c^2 \quad \text{.......... (4.34)}$$

or

Here, mc^2 is the total energy, $m_0 c^2$ is the rest mass energy and E_k is its kinetic energy. Hence, we write

$$E = E_k + m_0 c^2 \quad \text{.......... (4.35)}$$

and

$$E = mc^2 \quad \text{.......... (4.36)}$$

Equation (4.36) is known as ***Einstein's mass-energy relation***.

4.7 Important Points to Remembers

1. Space and Time transformation relations

	Galilean transformation	*Lorentz transformation*	*Inverse Lorentz transformation*
X - coordinates	$x' = x - vt$	$x' = \frac{x - vt}{\sqrt{1-(v^2/c^2)}}$	$x = \frac{x' - vt}{\sqrt{1-(v^2/c^2)}}$
Y - coordinates	$y' = y$	$y' = y$	$y = y'$
Z - coordinates	$z' = z$	$z' = z$	$z = z'$
Time coordinate	$t' = t$	$t' = \frac{t - (vx/c^2)}{\sqrt{1-(v^2/c^2)}}$	$t = \frac{t' + (vx'/c^2)}{\sqrt{1-(v^2/c^2)}}$

2. Time dilation : $T = \frac{T_0}{\sqrt{1-(v^2/c^2)}}$

3. Length contraction : $L = L_0 \sqrt{1-(v^2/c^2)}$

4. Einstein's mass energy relation : $E = mc^2$ and $E_k = mc^2 - m_0 c^2$.

4.8 Problems

Problem 1

A passenger in a train moving at 30 mps passes a man standing on a station platform at t = t ' = 0. Twenty second after the train passes the station, the man on the platform determines that a bird flying along the tracks in the same directions as the train is 800 m away. Using Galilean transformation, find the coordinates of the bird as determined by the passenger.

Solution :

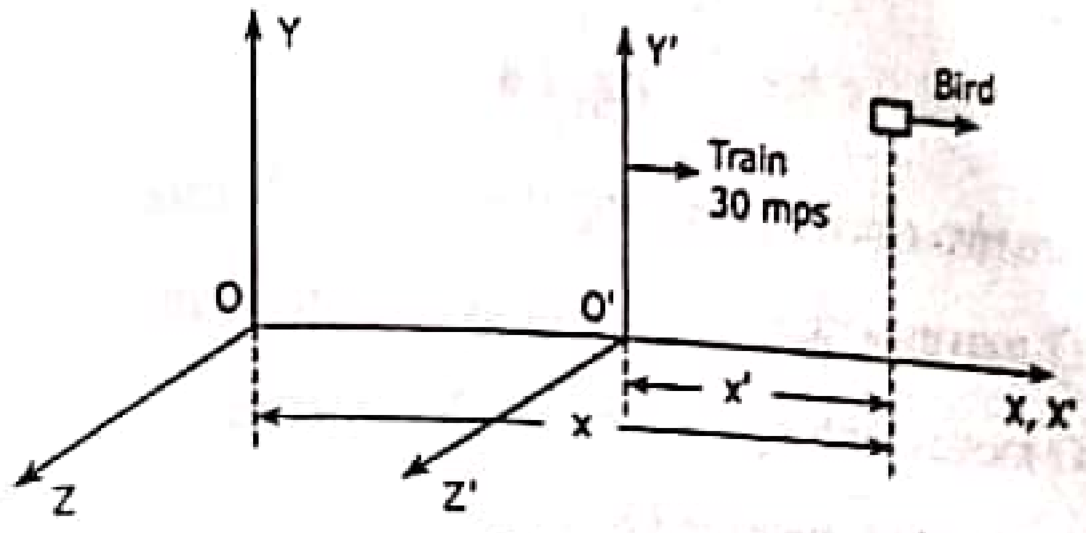


Fig. 4.5

By Galilean transformation $x' = x - vt$.

where, x = position of the bird on X-axis as seen by the stationary observer O on the station.

= 800 m

x ' = position of the bird on X-axis as seen by the passenger O' in the train

v = 30 mps = velocity of the train

t = 20 sec.

Hence, $x' = 800 - (30 \times 20) = 200$ m

For the stationary observer the coordinates of the bird is

$$(x, y, z, t) = (800 \text{ m}, 0, 0, 20 \text{ s})$$

For the passenger the coordinates of the bird is

$$(x', y', z', t) = (200 \text{ m}, 0, 0, 20 \text{ s})$$

Problem 2

A sample of radioactive material, at rest in the laboratory, ejects two electrons in opposite directions. One of the electrons has a speed of 0.6 c and the other has a speed of

0.7c as measured by a laboratory observer. According to Galilean transformation, what will be the speed of one electron as measured from the other? Comment on your result.

Solution :

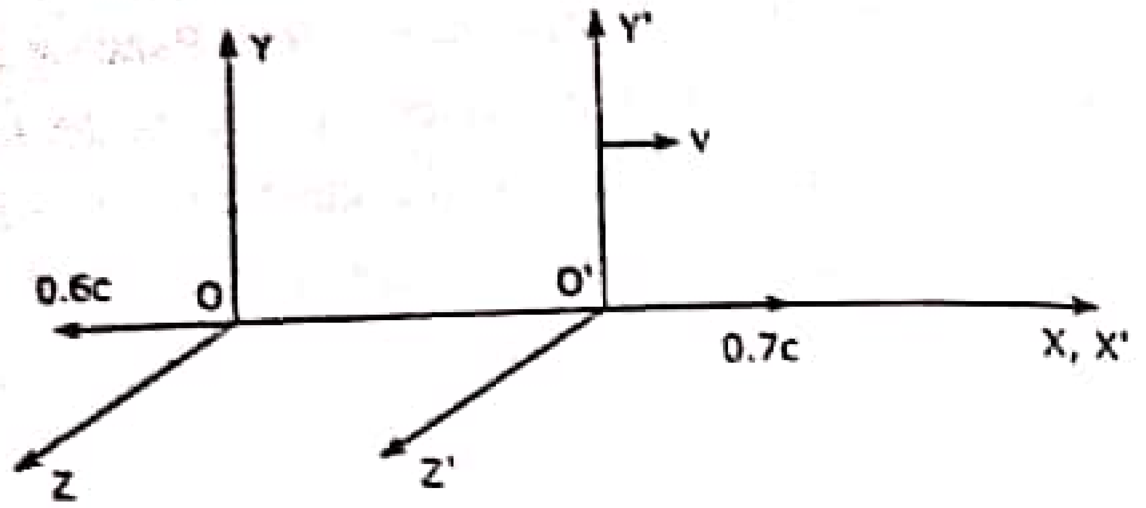


Fig. 4.6

Let O and O' are two electrons moving in –X and +X directions.

The electron O' moves with $v = 0.7c$ with respect to O in X direction.

The electron O moves with $u_x = -0.6c$ in –X direction.

Hence, the velocity of O' measured by O is

$$u_x' = u_x - v = -0.6c - 0.7c = -1.3c$$

The result shows a velocity greater than c by Galilean transformation. This is inconsistent with the special theory of relativity.

Problem 3

A train moving with a velocity of 60 kmph passes through a rail station at 12.00 clock. Twenty seconds later a bolt of lightening strikes the rail track one km away from the station in the direction of the train. Using Galilean transformation, find the coordinates of the lightening flash as measured by an observer at the station and by the engineer of the train.

Solution :

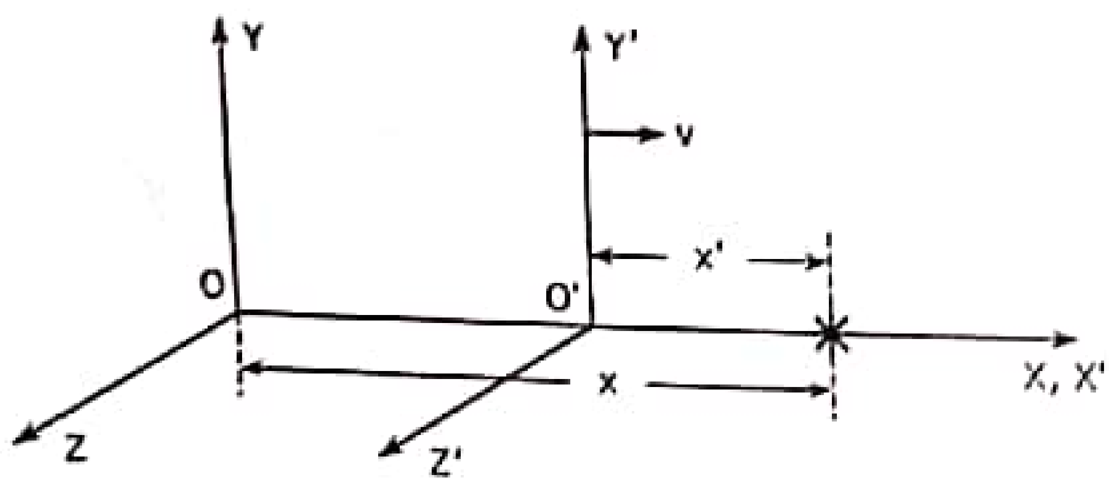


Fig. 4.7

The flash strikes the rail tract at

$$t = t' = \frac{20}{3600} = \frac{1}{180} \text{ hr}$$

Here O is the observer at the station and O' is the engineer in the train moving at v = 60 kmph with respect to O.

The observer at the station measures the x-coordinate of the flash as $x = 1$ km.

The engineer in the train measures the x-coordinates of the flash as

$$x' = x - vt$$

$$x' = 1 - \left(60 \times \frac{1}{180}\right) = \frac{2}{3} \text{ km} = 0.666 \text{ km}$$

Hence, the coordinates of the flash measured by the observer at the station is $\left(1 \text{ km}, 0, 0, \frac{1}{180} \text{ hr}\right)$ and by the engineering the train are $\left(0.666 \text{ km}, 0, 0, \frac{1}{180} \text{ hr}\right)$.

Problem 4

An event occurs at $x = 100$ m, $y = 10$ m, $z = 5$ m and $t = 1 \times 10^{-4}$ sec in a frame S. Find the coordinates of this event in a frame S' which is moving with a velocity 2.7×10^8 m/sec with respect to the frame S along the common XX' axes using (i) Galilean transformation and (ii) Lorentz transformation.

Solution :

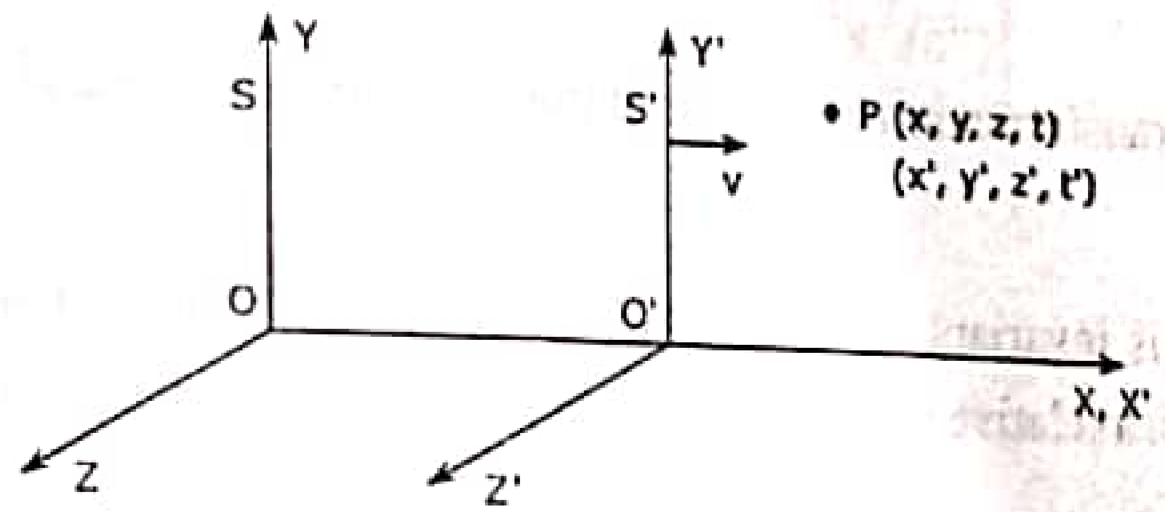


Fig. 4.8

P is the event with coordinates (x, y, z, t) in the stationary frame S and (x', y', z', t) in the frame S' moving with $v = 2.7 \times 10^8$ m/sec. with respect to S.

$$x = 100 \text{ m}, \; y = 10 \text{ m}, \; z = 5 \text{ m}, \; t = 10^{-4} \text{ sec.}$$

(i) According to Galilean transformation

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$

$$x' = 100 - (2.7 \times 10^8)\, 10^{-4} = -26900 \text{ m}$$

$y' = 10$ m, $z' = 5$ m, $t = 10^{-4}$ sec.

So the coordinates of the event are (-26900 m, 10 m, 5 m, 10^{-4} sec.)

(ii) According to Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1-(v^2/c^2)}},\ y' = y,\ z' = z,\ t' = \frac{t-(vx/c^2)}{\sqrt{1-(v^2/c^2)}}$$

Here, $$\sqrt{1-(v^2/c^2)} = \sqrt{1-\left(\frac{2.7\times10^8}{3\times10^8}\right)^2} = \sqrt{1-(0.9)^2} = 0.43588$$

$$\therefore \quad x' = \frac{100-(2.7\times10^8)(10^{-4})}{0.43588} = 6.712 \text{ m}$$

$$y' = 10 \text{ m}$$

$$z' = 5 \text{ m}$$

$$t' = \frac{t-(vx/c^2)}{0.43588} = \frac{10^{-4}-(2.7\times10^8)(100)/(3\times10^8)^2}{0.43588}$$

$$\therefore \quad t' = 2.2735\times10^{-4} \text{ sec}$$

Hence, the coordinates of the event in frame S' are

(61712 m, 10 m, 5 m, 2.2735×10^{-4} S).

Problem 5

Use Lorentz transformation to show that the quantity $(x^2+y^2+z^2-c^2t^2)$ is invariant.

Solution :

If a quantity is invariant it remains the same in all frames of reference. Consider frames S and S' with a relative velocity v. P is an event with coordinates (x, y, z, t) in S and (x', y', z', t') in S'.

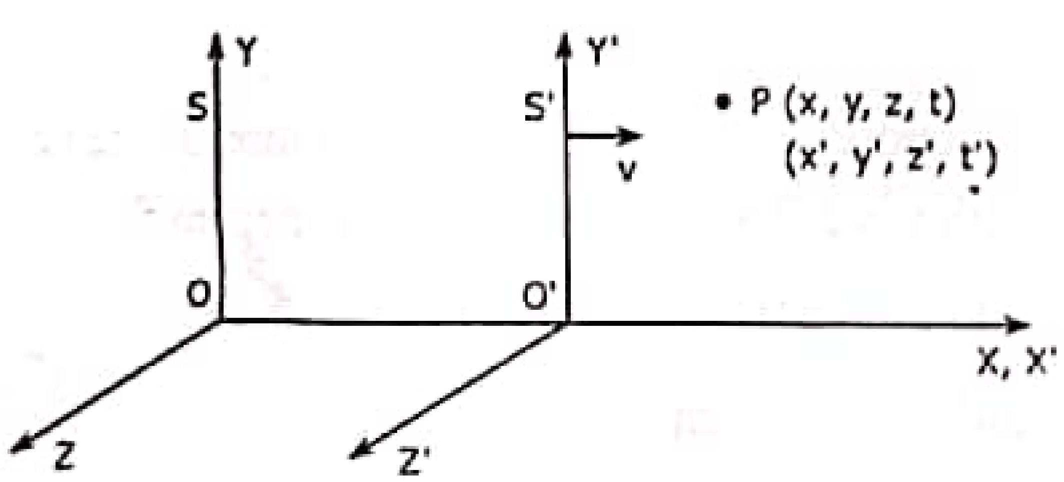


Fig. 4.9

According to Lorentz transformation, we have

$$x' = \frac{x - vt}{\sqrt{1-(v^2/c^2)}},\ y' = y,\ z' = z,\ t' = \frac{t-(vx/c^2)}{\sqrt{1-(v^2/c^2)}}$$

If $(x^2+y^2+z^2-c^2t^2)$ is invariant, we should have

$$(x^2+y^2+z^2-c^2t^2) = x'^2+y'^2+z'^2-c^2t'^2$$

$$\text{R.H.S.} = x'^2+y'^2+z'^2-c^2t'^2$$

$$= \left[\frac{x-vt}{\sqrt{1-(v^2/c^2)}}\right]^2 + y^2+z^2-c^2\left[\frac{t-(vx/c^2)}{\sqrt{1-(v^2/c^2)}}\right]^2$$

$$= \frac{(x-vt)^2-c^2[t-(vx/c^2)]^2}{1-(v^2/c^2)}+y^2+z^2$$

$$= \frac{x^2-2x\,vt+v^2t^2-c^2\left(t^2-2\frac{vxt}{c^2}+\frac{v^2x^2}{c^4}\right)}{1-(v^2/c^2)}+y^2+z^2$$

$$= \frac{x^2-2x\,vt+v^2t^2-c^2t^2+2x\,vt-\frac{v^2x^2}{c^2}}{1-(v^2/c^2)}+y^2+z^2$$

$$= \frac{x^2\left[1-(v^2/c^2)\right]-c^2t^2\left[1-(v^2/c^2)\right]}{1-(v^2/c^2)}+y^2+z^2$$

$$= \frac{x^2-c^2t^2\left[1-(v^2/c^2)\right]}{1-(v^2/c^2)}+y^2+z^2$$

$$= x^2+y^2+z^2-c^2t^2$$

$$\text{R.H.S.} = \text{L.H.S.}$$

Hence, proved.

Problem 6

Using Lorentz transformation, show that the circle $x^2+y^2=a^2$ in frame S appears to be an ellipse in frame S ' moving with a velocity v with respect to S.

Problem 8

The length of a rod is found to be half of its length when at rest. What is the speed of the rod relative to the observer?

Solution :

Data : $L = \frac{L_0}{2}$

Formula : $L = L_0 \sqrt{1-(v^2/c^2)}$

$$\frac{L_0}{2} = L_0 \sqrt{1-(v^2/c^2)}$$

$$\sqrt{1-(v^2/c^2)} = \frac{1}{2} \qquad \therefore \quad 1-\frac{v^2}{c^2} = \frac{1}{4}$$

$$\frac{v^2}{c^2} = \frac{3}{4} \qquad \therefore \quad v = \frac{\sqrt{3}}{2} c = 0.866 \text{ c}$$

.... **Ans.**

Problem 9

A 1 m long rod is moving along its length with a velocity 0.6 c. Calculate its length as it appears to an observer on the earth.

Solution :

Data : $v = 0.6$ c, $L_0 = 1$ m.

Formula : $L = L_0 \sqrt{1-(v^2/c^2)}$

Calculations : $L = 1\sqrt{1-(0.6)^2} = 0.8$ m

.... **Ans.**

Problem 10

A rod has a length of 2 m. Find its length when it is carried in a rocket with a speed of 0.9 c.

Solution :

Data : $L_0 = 2$ m, $v = 0.9$ c

Formula : $L = L_0 \sqrt{1-(v^2/c^2)}$

Calculations : $L = 2\sqrt{1-(0.9)^2}$

$\therefore$ $L = 0.872$ m

.... **Ans.**

Problem 11

A rocket ship is 100 m long on the ground. When it is in flight, its length is 99 m to an observer on the ground. What is its speed?

Solution :

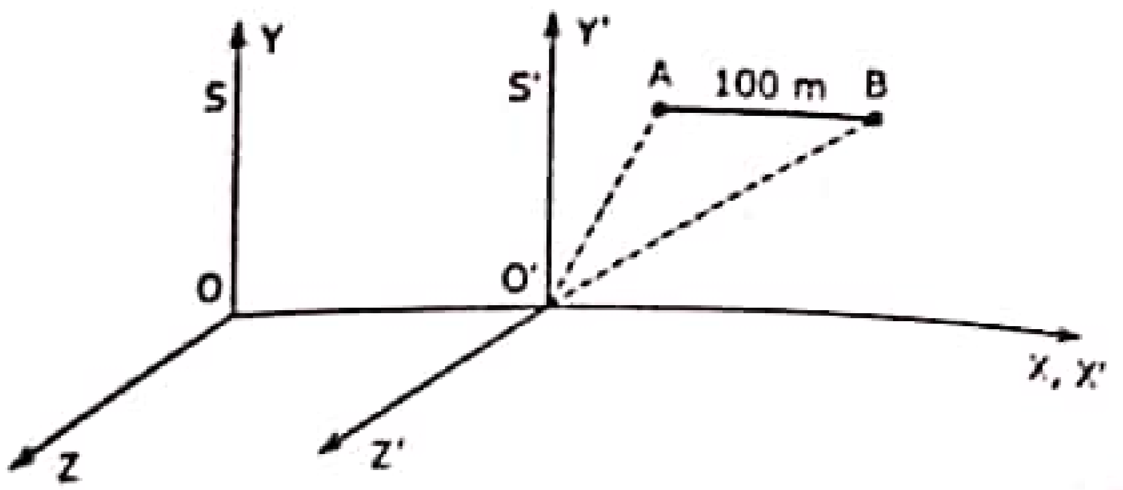


Fig. 4.10

Data : Observer O' in the rocket moves with a velocity v with respect to the observer O on the ground. Hence,

$l' = 100$ m as observed by O' and $l = 99$ m as observed by O.

Formula : $l = l'\sqrt{1-(v^2/c^2)}$

Calculations : $l^2 = l'^2[1-(v^2/c^2)]$

$$\left(\frac{l}{l'}\right)^2 = 1-\frac{v^2}{c^2}$$

$$v^2 = c^2\left(1-\frac{l^2}{l'^2}\right)$$

$$v = c\sqrt{1-\left(\frac{l}{l'}\right)^2} = 3\times10^8\sqrt{1-\left(\frac{99}{100}\right)^2}$$

$\therefore$ $v = 4.23\times10^7$ m/sec. .... Ans.

Problem 12

Calculate the percentage contraction in the length of a rod moving with a speed of 0.8 c in a direction at an angle of 60° with its own length.

Solution :

Data : $v = 0.8\,c,\ \theta = 60°$

Formula : $l = l_0\sqrt{1-(v^2/c^2)}$

Calculations : θ is the angle between frames S and S'. The rod AB lies in the XY plane in frame S. As it travels in S' at an angle of 60° with frame S.

$l_x = l\cos 60°,\quad l_y = l\sin 60°.$

$$l_x' = l_x\sqrt{1-(v^2/c^2)}$$

$$= l\cos 60°\sqrt{1-(0.8)^2}$$

$$\therefore\quad l_x' = 0.3\,l$$

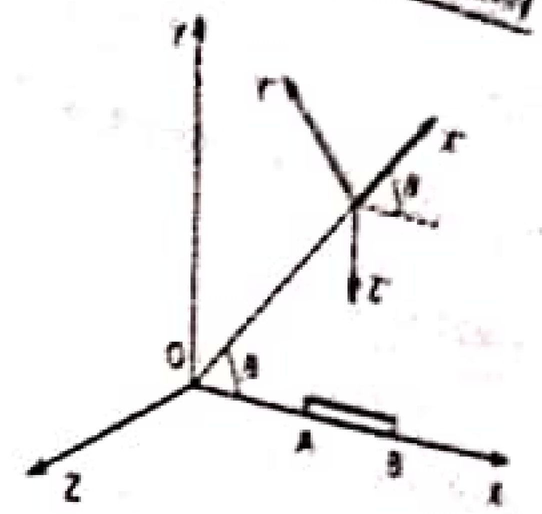


Fig. 4.11

$$l_y' = l_y = 1\sin 60° = \frac{\sqrt{3}\,l}{2}$$

as there is no motion perpendicular to the length of the rod.

Hence, $$l' = \sqrt{l_x'^2 + l_y'^2} = \sqrt{(0.3l)^2+\left(\frac{\sqrt{3}l}{2}\right)^2}$$

$$\therefore\quad l' = 0.9165\,l$$

Percentage contraction $= \dfrac{l-l'}{l}\times 100$

$$= \frac{l-0.9165l}{l}\times 100 = 8.2\,\%$$

Problem 13

In the laboratory, the lifetime of a particle moving with speed 2.8×10^8 m/sec is found to be 2×10^{-7} sec. Calculate the proper life time of the particle.

Solution :

Data : $v = 2\times10^{-7}$ sec, $T = 2.8\times10^8$ m/sec.

Formula : $T = \dfrac{T_0}{\sqrt{1-(v^2/c^2)}}$; T = Life time measured, T_0 = Proper lifetime.

Calculations :

$$T_0 = T\sqrt{1-(v^2/c^2)}$$

$$= 2\times10^{-7}\sqrt{1-\left(\frac{2.8\times10^8}{3\times10^8}\right)^2}$$

$\therefore \quad T_o = 7.18 \times 10^{-7}$ sec.

Ans. : Proper life time = 7.18×10^{-7} sec.

Problem 14

A certain process requires 10^{-6} sec. to occur in an atom at rest in laboratory. How much time will this process require to an observer in the laboratory, when the atom is moving with a speed of 5×10^7 m/sec ?

Solution :

Data : $T_o = 10^{-6}$ sec., $v = 5 \times 10^7$ m/sec.

Formula : $$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations : $$T = \frac{10^{-6}}{\sqrt{1-\left(\frac{5\times10^7}{3\times10^8}\right)^2}} = 1.014 \times 10^{-6} \text{ sec.}$$

Ans. : Time = 1.014×10^{-6} sec.

Problem 15

The mean life of a meson is 2×10^{-8} sec. Calculate the mean life of a meson moving with a velocity of 0.8 c.

Solution :

Data : $T_o = 2 \times 10^{-8}$ sec., $v = 0.8$ c.

Formula : $$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations : $$T = \frac{2\times10^{-8}}{\sqrt{1-\left(\frac{0.8\,c}{c}\right)^2}} = \frac{2\times10^{-8}}{\sqrt{1-(0.8)^2}}$$

$\therefore \quad T = 3.33 \times 10^{-8}$ sec.

Ans. : Mean life of a meson = 3.33×10^{-8} sec.

Problem 16

What is the velocity of π mesons whose observed mean life is 2.5×10^{-7} sec. The proper mean life of these π mesons is 2.5×10^{-8} sec.

Solution :

Data : $T_o = 2.5 \times 10^{-8}$ sec., $T = 2.5 \times 10^{-7}$ sec.

Formula : $$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations :

$$\sqrt{1-(v^2/c^2)} = \frac{T_o}{T}$$

$$v^2 = c^2\left[1-\left(\frac{T_o}{T}\right)^2\right]$$

$$v^2 = c^2\left[1-\left(\frac{2.5\times10^{-8}}{2.5\times10^{-7}}\right)^2\right]$$

$\therefore \quad v = 0.995$ c .... Ans.

Problem 17

A clock keeps correct time on the earth. It is put on the space ship moving uniformly with a speed of 10^8 m/sec. How many hours does it appear to lose per day?

Solution :

Data : T = 24 hrs as measured in the space ship

T_o = the time observed by an observe on the earth.

$v = 10^8$ m/sec.

Formula : $$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations : $$T_o = T\sqrt{1-\frac{v^2}{c^2}} = 24\sqrt{1-\left(\frac{10^{-8}}{3\times10^{-8}}\right)^2}$$

$$\therefore \quad T_o = 24 \times \frac{2\sqrt{2}}{3} = 22.63 \text{ sec.}$$

.... Ans.

Time lost per day = 24 − 22.63 = 1.37 hr

Problem 18

With what velocity should a rocket move, so that every year spent on it corresponds to 4 years on the earth?

Solution :

Data : T_o = correct time = 1 year on the rocket

$T = 4$ years, as appears from the earth

Formula : $$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations :

$$1 - \frac{v^2}{c^2} = \frac{T_o}{T}$$

$$v^2 = c^2\left[1-\left(\frac{T_o}{T}\right)^2\right] = c^2\left[1-\left(\frac{1}{4}\right)^2\right]$$

$$\therefore \quad v = 0.97\ c$$

.... Ans.

Problem 19

With what velocity should a space ship fly so that every day spent on it may correspond to three days on the earth's surface.

Solution :

Data : $T = 3$ days as it appears on the earth's surface.

$T_o = 1$ day as measured in the spaceship.

Formula : $$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations :

$$1 - \frac{v^2}{c^2} = \left(\frac{T_o}{T}\right)^2$$

$$\therefore \quad v^2 = c^2\left[1-\left(\frac{T_o}{T}\right)^2\right] = c^2\left[1-\frac{1}{9}\right]$$

$$\therefore \quad v = \frac{2\sqrt{2}}{3}\ c = 0.47\ c$$

.... Ans.

Problem 20

At what speed should a clock be moved so that it may appear to lose 1 minute in each hour?

Solution :

Data : The clock, loses 1 min in each hour. So it must record 59 min. for each hour. Hence, $T_o = 59$ min., $T = 1$ hour $= 60$ min.

Formulae : $$T = \frac{T_o}{\sqrt{1-(v^2/c^2)}}$$

$$\therefore \quad \sqrt{1-\frac{v^2}{c^2}} = \frac{T_o}{T}$$

$$\therefore \quad v^2 = c^2\left[1-\left(\frac{T_o}{T}\right)^2\right] = c^2\left[1-\left(\frac{59}{60}\right)^2\right]$$

$$\therefore \quad v = 0.1818\ c$$

.... Ans.

Problem 21

At what velocity will the mass of a body is 2.25 times its rest mass?

Solution :

Data : $m = 2.25\ m_o$

Formula : $$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations :

$$1 - \frac{v^2}{c^2} = \left(\frac{m_o}{m}\right)^2$$

$$v^2 = c^2\left[1-\left(\frac{m_o}{m}\right)^2\right] = c^2\left[1-\left(\frac{1}{2.25}\right)^2\right]$$

$$\therefore \quad v = 0.895\ c$$

.... Ans.

Problem 22

With what velocity a particle should move so that its mass appears to increase by 20 % of its rest mass?

Solution :

Data : $m = m_o + 20\,\%\, m_o = 1.2\, m_o$

Formula : $$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations : $$\frac{m}{m_o} = \frac{1}{\sqrt{1-(v^2/c^2)}}$$

$$1 - \frac{v^2}{c^2} = \left(\frac{m_o}{m}\right)^2$$

$$v^2 = c^2\left[1-\left(\frac{m_o}{m}\right)^2\right] = c^2\left[1-\left(\frac{1}{1.2}\right)^2\right]$$

$\therefore$ $$v^2 = c^2\,[0.30558]$$

$\therefore$ $$v = 0.553\ c$$ .... Ans.

Problem 23

If the kinetic energy of a body is double its rest mass energy calculate its velocity.

Solution :

Data : $E_k = 2\, m_o\, c^2$

Formulae : $E = E_k + m_o\, c^2,\ E = mc^2,\ m = \dfrac{m_o}{\sqrt{1-(v^2/c^2)}}$

Calculations : $$mc^2 = 2\, m_o\, c^2 + m_o\, c^2$$

$$m = 3\, m_o$$

$\therefore$ $$\frac{m_o}{\sqrt{1-(v^2/c^2)}} = 3\, m_o$$

$$1 - \frac{v^2}{c^2} = \frac{1}{9} \quad \therefore \quad v^2 = c^2\,\frac{8}{9}$$

$\therefore$ $$v = \frac{2\sqrt{2}}{3}\, c = 0.94\ c$$ .... Ans.

Problem 24

The mass of a moving electron is 11 times its rest mass. Calculate its kinetic energy and momentum.

Solution :

Data : $m = 11\, m_o$

Formulae : $E = E_k + m_o\, c^2,\ E = mc^2,\ m = \dfrac{m_o}{\sqrt{1-(v^2/c^2)}}$

Calculations : $$E_k = E_k + m_o\, c^2$$

$$= 11\, m_o\, c^2 - m_o\, c^2 = 10\, m_o\, c^2$$

$$= 10 \times 9.1 \times 10^{-31} \times (3 \times 10^8)^2\ J$$

$$= 8.2 \times 10^{-13}\ J = \frac{8.2 \times 10^{-13}}{1.6 \times 10^{19}}\ eV$$

$\therefore$ $$E_k = 5.1\ MeV$$

Now, $$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}} \quad \therefore \quad \frac{1}{\sqrt{1-(v^2/c^2)}} = \frac{m}{m_o} = 11$$

$$1 - \frac{v^2}{c^2} = \left(\frac{1}{11}\right)^2 \quad \therefore \quad \frac{v^2}{c^2} = 1 - \frac{1}{121}$$

$\therefore$ $$v = 2.98 \times 10^8\ m/sec$$

$$p = \frac{m_o\, v}{\sqrt{1-(v^2/c^2)}} = 11\, m_o\, v$$

$$= 11 \times 9.1 \times 10^{-31} \times 2.98 \times 10^8$$

$\therefore$ $$p = 2.98 \times 10^{-21}\ N\text{-}sec.$$ .... Ans.

Problem 25

How fast must an electron move in order to have its mass equal to the rest mass of the proton (1.67×10^{-27} kg)?

Solution :

Data : $m = m_p =$ rest mass of a proton $= 1.67 \times 10^{-27}$ kg

$m_o = 9.1 \times 10^{-31}$ kg

Formula : $$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations :

$$1-\frac{v^2}{c^2} = \left(\frac{m_o}{m}\right)^2$$

$$v^2 = c^2\left[1-\left(\frac{m_o}{m}\right)^2\right] = c^2\left[1-\left(\frac{9.1\times10^{-31}}{1.67\times10^{-27}}\right)^2\right]$$

$$\therefore \quad v = 0.9985\ c \quad \text{.............. Ans}$$

Problem 26

Kinetic energy of a particle is (i) 3 times, (ii) equal to its rest mass energy. What is its velocity?

Solution :

Data : (i) $E_k = 3\, m_o c^2$, (ii) $E_k = m_o c^2$.

Formulae : $E = E_k + m_o c^2$, $E_k = mc^2 - m_o c^2$, $m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$

Calculations : (i)

$$E_k = mc^2 - m_o c^2$$

$$3\, m_o c^2 = mc^2 - m_o c^2$$

$$m = 4\, m_o$$

$$\frac{m_o}{\sqrt{1-(v^2/c^2)}} = 4\, m_o \qquad \therefore \quad 1-\frac{v^2}{c^2} = \frac{1}{16}$$

$$v^2 = c^2\left(1-\frac{1}{16}\right) \qquad \therefore \quad v = 0.968\ c \quad \text{.............. Ans}$$

(ii)

$$E_k = mc^2 - m_o c^2$$

$$m_o c^2 = mc^2 - m_o c^2$$

$$mc^2 = 2\, m_o c^2$$

$$\therefore \quad m = 2\, m_o \qquad \therefore \quad \frac{m_o}{\sqrt{1-(v^2/c^2)}} = 2\, m_o$$

$$1-\frac{v^2}{c^2} = \frac{1}{4}$$

$$v^2 = c^2\left(1-\frac{1}{4}\right) \qquad \therefore \quad v = 0.866\ c \quad \text{.............. Ans}$$

Problem 27

Find the velocity of a 0.1 MeV electron according to classical and relativistic mechanics.

Solution :

Data : $E_k = 0.1\ \text{MeV} = 0.1\times10^6\times1.6\times10^{-19} = 1.6\times10^{-14}$ Joules

$m = 9.1\times10^{-31}$ kg (classically),

$m_o = 9.1\times10^{-31}$ kg (relativistically)

Formulae : $E_k = \frac{1}{2}mv^2$ in classical mechanics and

$E_k = mc^2 - m_o c^2$ in relativistic mechanics.

$$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$$

Calculations : In classical mechanics,

$$v = \sqrt{\frac{2}{m}E_k} = \sqrt{\frac{2}{9.1\times10^{-31}}\times1.6\times10^{-14}}$$

$$\therefore \quad v = 1.87\times10^8\ \text{m/sec}$$

In relativistic mechanics,

$$E_k = \frac{m_o c^2}{\sqrt{1-(v^2/c^2)}} - m_o c^2$$

$$E_k = m_o c^2\left(\frac{1}{\sqrt{1-(v^2/c^2)}}-1\right)$$

$$\frac{E_k}{m_o c^2} = \frac{1}{\sqrt{1-(v^2/c^2)}} - 1$$

$$\frac{1}{\sqrt{1-(v^2/c^2)}} = \frac{E_k}{m_o c^2} + 1$$

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{1+\left(\frac{E_k}{m_o c^2}\right)} \qquad \therefore \quad 1-\frac{v^2}{c^2} = \frac{1}{\left(1+\frac{E_k}{m_o c^2}\right)^2}$$

$$\frac{v^2}{c^2} = 1 + \frac{1}{\left(1+\frac{E_k}{m_o c^2}\right)^2}$$

$$\therefore \quad v^2 = c^2\left[1+\frac{1}{\left(1+\frac{E_k}{m_o c^2}\right)^2}\right]$$

$$= c^2\left[1+\frac{1}{\left(1+\frac{1.6\times10^{-14}}{9.1\times10^{-31}\times3\times10^8}\right)^2}\right]$$

$$\therefore \quad v = 0.54\ c$$

.... Ans.

4.9 Review Questions

(A) Short answer type :

1. Distinguish between the Special Theory of Relativity and the Classical Theory of Relativity.
2. Define an inertial frame of reference.
3. What are non-inertial frames of reference?
4. What are Galilean transformation?
5. Write Lorentz transformation equation.
6. Write inverse Lorentz transformation equations.
7. What are the postulates of Special theory of relativity?
8. Explain time dilation and length contraction.
9. Show how does the mass of an object vary with velocity.
10. What are (i) proper length and (ii) proper time?

(B) Long answer type :

1. Derive Galilean transformation equations for (i) position, (b) velocity, and (c) acceleration.

MODULE 5

Nanotechnology

(**Prerequisites :** Scattering of electrons, Tunneling effect, Electrostatic focusing, Magneto static focusing.)

Nanomaterials : Properties (Optical, electrical, magnetic, structural, mechanical) and applications, Surface to volume ratio, Two main approaches in nanotechnology - Bottom up technique and Top down technique.

Tools for characterization of Nanoparticles : Scanning Electron Microscope (SEM), Transmission Electron Microscope (TEM), Atomic Force Microscope (AFM). Methods to synthesize Nanomaterials : Ball milling, Sputtering, Vapour deposition, Solgel. **(04 Hours) (Weightage - 10 %)**

Course Outcome : CO5 : Learner will be able to illustrate the knowledge of synthesis, characterisation and applications of nanomaterials.

SYNOPSIS

5.1 Introduction

5.2 Prerequisite

5.3 Nanomaterials

5.4 Tools for Characterization of Nanoparticles

5.5 Methods to Synthesize

5.6 Applications of Nanomaterials

Important Points to Remember

Exercise

Previous University Examination Questions with Solutions

5.1 Introduction

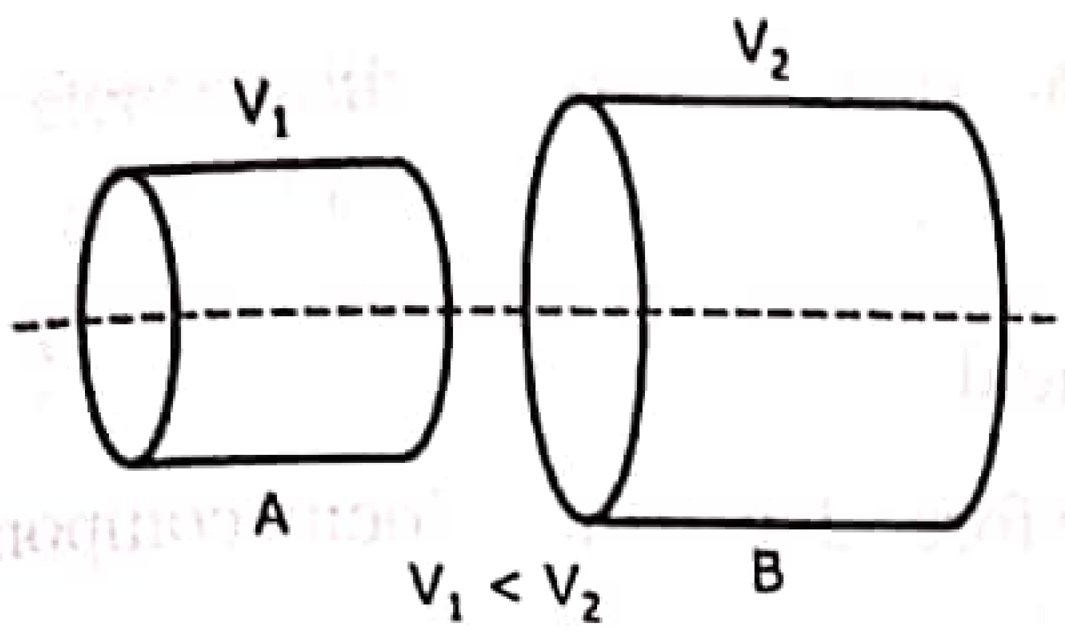
Nanotechnology is the term given to those areas of science and engineering where phenomena that take place at dimensions in the nanometer scale are utilised in the design, characterisation, production and application of materials, structure, devices and systems. Nanotechnology is the manipulation of matter on an atomic, molecular and supermolecular scale and deals with various structures of matter having dimensions of the order of 100 nm.

5.2 Prerequisite

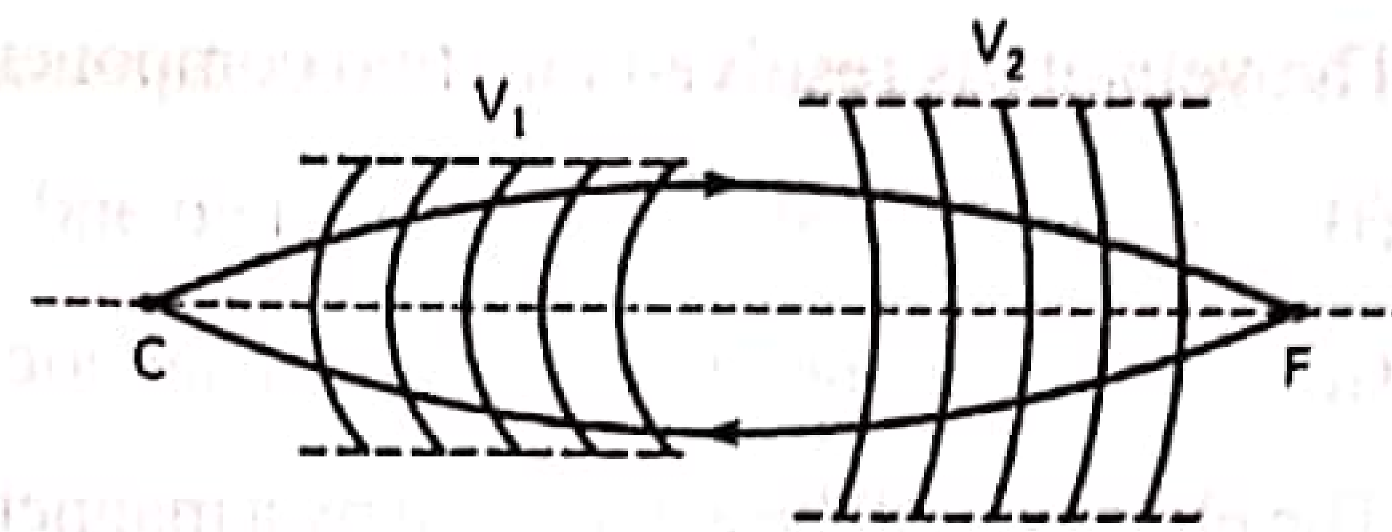
5.2.1 : Electrostatic Focussing

An electric field can be represented by a series of very closed spaced imaginary surfaces on which at every point the electric potential is constant. Such surfaces are called equipotential surfaces. The electric field is always directed perpendicularly to the equipotential surface at every point on it.

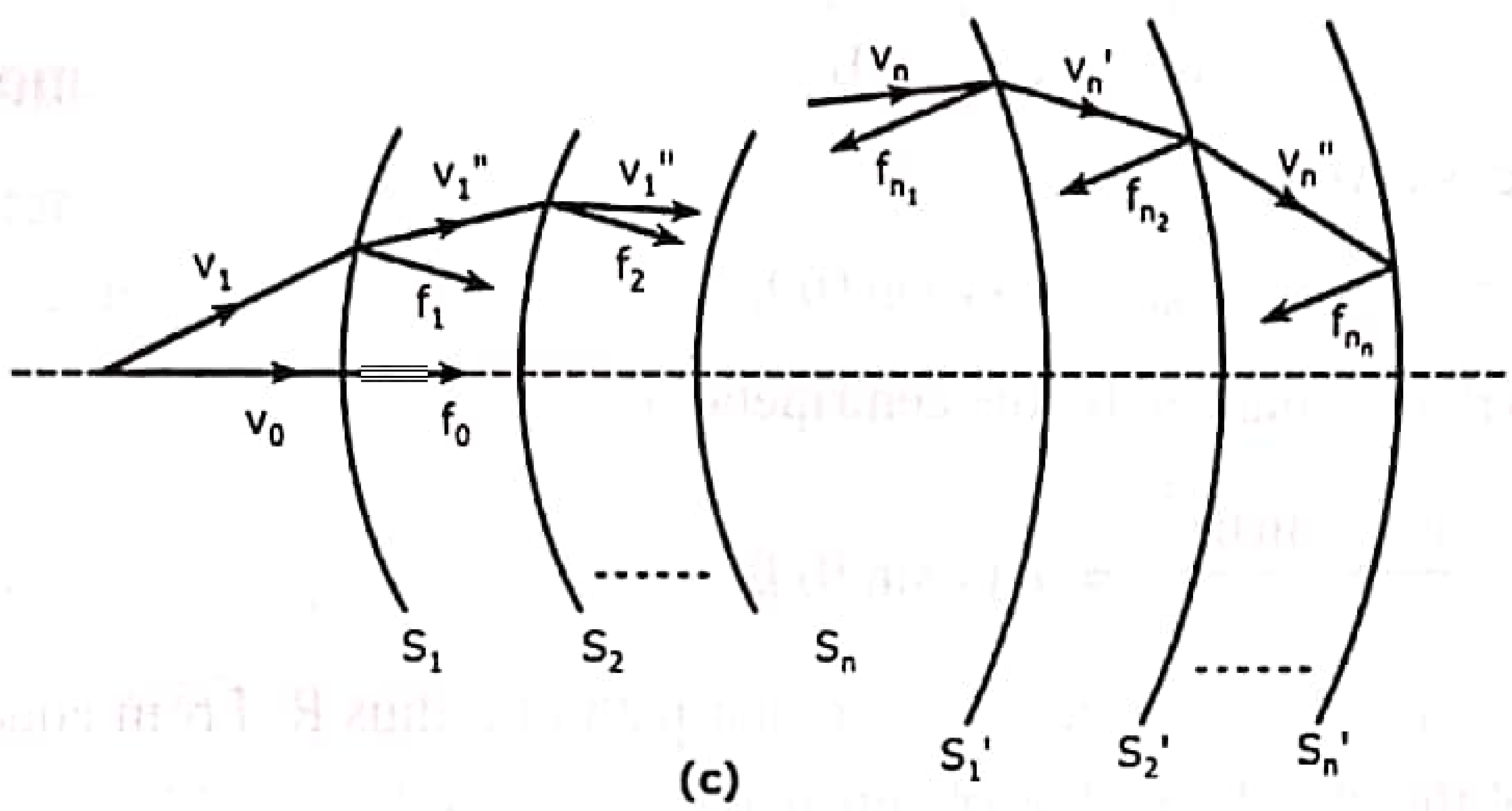
- An electron lens consists of two coaxial metallic cylinders A and B separated by some distance. The cylinders A and B are maintained at different positive potentials V_1 and V_2 respectively such that $V_2 > V_1$. These positively charged cylinders are called ***cylindrical anodes*** as shown in Fig. 5.1 (a).
- In Fig. 5.1 (b), the equipotential surfaces of the two cylinders are shown. The electron beam undergoes bending at each equipotential surface and finally it is focussed at the point F. The gradual bending of the electron beam is illustrated in Fig. 5.1 (c).
- Consider the electron travelling with velocity v_o along the axis of the system. The electric force f_o acting perpendicular to the equipotential surfaces drags the electron along the axis to the point P.
- A more deflected electron travelling with a velocity v_1 is bent by the perpendicular electric forces f_1 at the first equipotential surface S_1, f_2 at the second surface and so on. At every equipotential surface the velocity changes its direction.
- A highly deviated electron is collected by the cylinder B. Its velocity v_n changes to v_n' to v_n'' and so on at consecutive equipotential surfaces, S_1', S_2', ..., S_n' due to the perpendicular forces f_{n_1}, f_{n_2}, ..., f_{n_n} respectively. Thus electrons emitted by the cathode, C in various directions are focused at point F.



(a) Cylindrical anodes



(b) Electrostatic focusing of electrons



(c)

Fig. 5.1 : Electrostatic focussing

5.2.2 : Magnetostatic Focussing

- A uniform magnetic field has a focussing effect on an electron beam.
- Consider an electron beam originating at point O. In the beam electrons travel in different directions.
- Let an electron travelling with velocity v enter an uniform magnetic field making an angle θ.

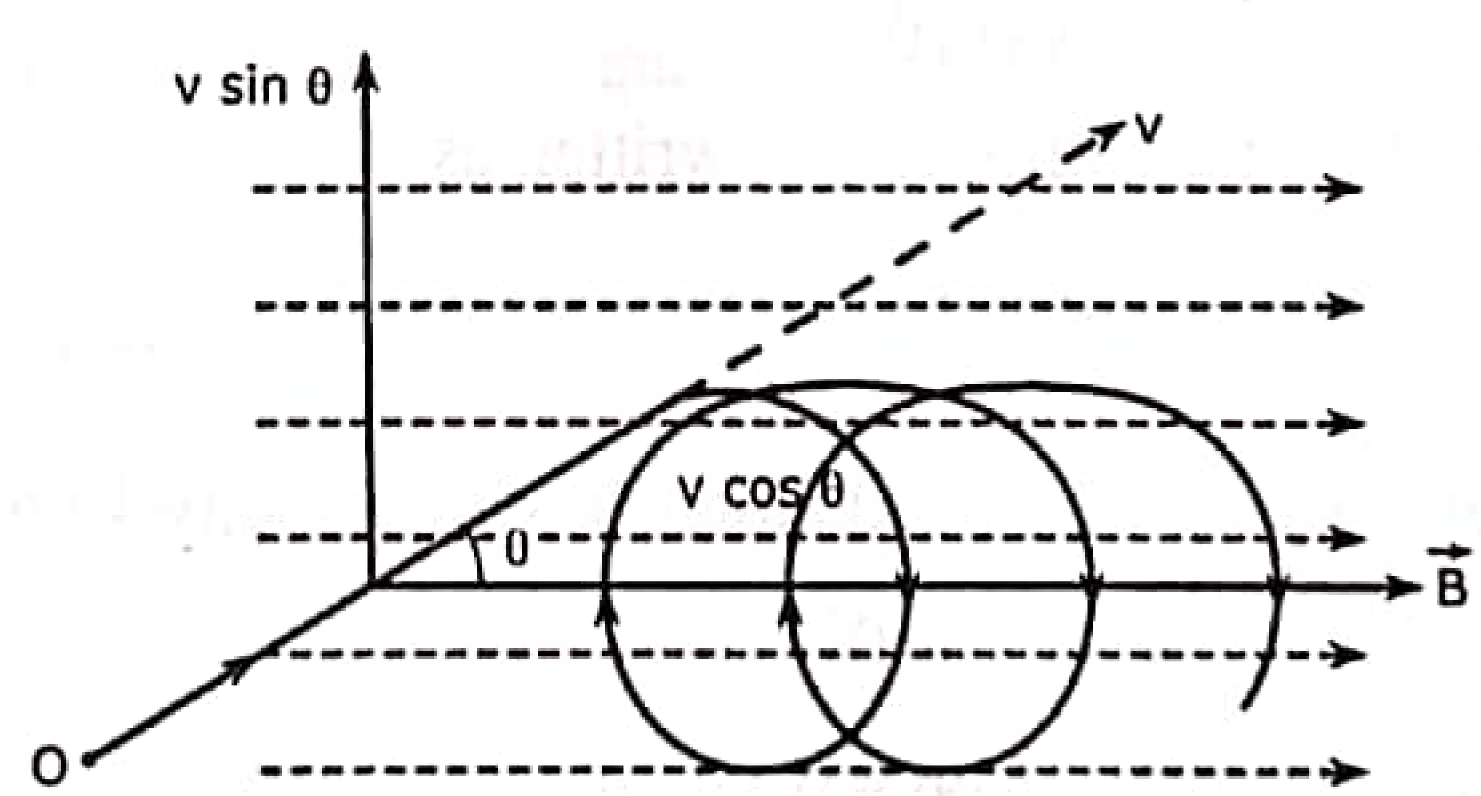


Fig. 5.2 : Magnetostatic Focussing

- The velocity is resolved into two components.

  (i) v cos θ along the magnetic field and

  (ii) v sin θ perpendicular to the magnetic field.

- The electron will be acted upon by a magnetic force due to the velocity component v sin θ. According to equation

$$\vec{F} = e\,\vec{v} \times \vec{B} \qquad \text{.............. (5.1)}$$

the force is given by

$$f_m = e\,(v \sin\theta)\,B \qquad \text{.............. (5.2)}$$

which will be balanced by the centripetal force as

$$\frac{m(v\sin\theta)^2}{R} = e\,(v \sin\theta)\,B \qquad \text{.............. (5.3)}$$

and the electron will trace out a circular path of radius R. From equation (5.2) the radius of the circular path is obtained as

$$R = \frac{mv}{eB}\sin\theta \qquad \text{.............. (5.3)}$$

From equation (5.1) it is obvious that

- There will be no force acting on the electron due to the component v cos θ. However, there will be a translational motion of the electron due to v cos θ.
- The time taken by the electron to complete one circular motion is its time period, given by

$$T = \frac{2\pi R}{v\sin\theta}$$

Using equation (5.3) here, this can be written as

$$T = \frac{2\pi m}{eB} \qquad \text{.............. (5.4)}$$

- In T secs. the electron is translated along B with velocity v cos θ over a distance

$$p = (v\cos\theta)\,T$$

Using equation (5.4), this can be written as

$$p = \frac{2\pi m}{eB}\,v\cos\theta \qquad \text{.............. (5.5)}$$

- Hence with the two velocity components the electron traces out a helical path of radius R and linear velocity cos θ. The parameter p is called the ***pitch of the helix.***
- For small θ values, the pitch is constant,

$$p = \frac{2\pi m}{eB} \qquad \text{.............. (5.6)}$$

5.3 Nanomaterials

The materials with structural units which are an aggregate of atoms or molecules with dimensions in Nano scale *i.e.*, between 1 nm and 100 nm are called *nanomaterial.* Engineered nanomaterials are produced with required dimensions *i.e.*, either one or two or all the three dimensions in the nanoscale.

- Nano materials that have at least one dimension in the nano scale are called ***nanolayers***, such as thin films or surface coatings.
- If two dimensions of a nanomaterial are in the nano scale they are categorized as ***nanotubes or nanowires.***
- Lastly, nanomaterials that have all the three dimensions in the nano scale are called ***nanoparticles.***
- Nanomaterials made up of nanometer sized grains are called ***nanocrystalline solids.***

5.3.1 : Properties of Nano materials

The properties of nano materials are very different from those of the bulk materials. One important difference is the increased surface area to volume ratio of nanostructures. Nonstructures are also associated with quantum effects. These special properties are due to the size of the nano particles.

(a) Optical Properties

Depending upon their constituents, nanoparticles absorb a range of wavelengths and emit a characteristic wavelength. It is possible to alter the linear and non-linear optical properties by altering the crystals. Nonomaterials are, therefore, used in electrochromic devices.

When light is incident on a nanoparticle it can be scattered or absorbed. The total effect of scattering and absorption is referred to as extinction. Nanoparticles are in the size regime where the fraction of light that is scattered or absorbed can vary greatly depending on the particle diameter. At diameters less than 20 nm, nearly all of the extinction is due to absorption. At sizes above 100 nm, the extinction is mostly due to scattering. By designing a nanoparticle with desirable diameter the optimal amount of scattering and absorption can be achieved.

(b) Electrical Properties

The size of nanomaterials leads to an increase in their ionization potential. Due to quantum confinement the electronic bands come closer and become narrow. Energy states are transformed into localized molecular bonds which can be altered by the passage of current or by the application of a field. The change in electrical properties is material dependant. As an example, metals undergo an increase in conductivity whereas in the case of non-metallic nanomaterials a decrease in conductivity is observed.

(c) Magnetic Properties

Nanosized materials are more magnetic than their counterparts in the bulk. Nanoparticles of non-magnetic solids also may demonstrate magnetic properties.

The dynamics of magnetization and demangetization of magnetic materials in any device are governed by the presence of domain walls and regions with magnetization in different directions. In the case of magnetic nanoparticles, the magnetic vectors become aligned in the ordered pattern of a single domain in the presence of a DC magnetic field. In such cases, phenomena of thermal excitation or quantum mechanical tunnelling change the hysteresis loop of magnetic nanoparticles as compared to the bulk material.

(d) Mechanical and Structural Properties

Due to the formation of nanoparticles the atoms which are on the surface face different potentials in different directions. The resulting surface stress in nanoparticles modifies its mechanical and structural properties. The intrinsic elastic modulus of a nanostructured material is essentially the same as that of the bulk material having the micrometer sized grains until the grain size becomes very small, < 5 nm. If the grain size is below 20 nm the Young's modulus of the material begins to decrease from its value in conventional grain sized materials. Most nanostructured materials are quite brittle and display reduced ductility under tension.

In nanomaterials, because of their nanosize many of their mechanical properties are modified from its value in bulk materials. These properties among other are hardness and elastic modulus, fracture toughness, scratch resistance and fatigue strength. Energy dissipation, mechanical coupling and mechanical non-linearities are influenced by structuring components at the nanometer scale.

5.3.2 : Surface Area to Volume Ratio

The surface area to volume ratio determines the efficiency of the object. The surface area to volume ratio for a material or substance made of nanoparticles has a significant effect on the properties of the material. Nanomaterials have much greater surface area per unit volume ratio compared with the bulk materials.

Take for example, a cube with side length 'a'.

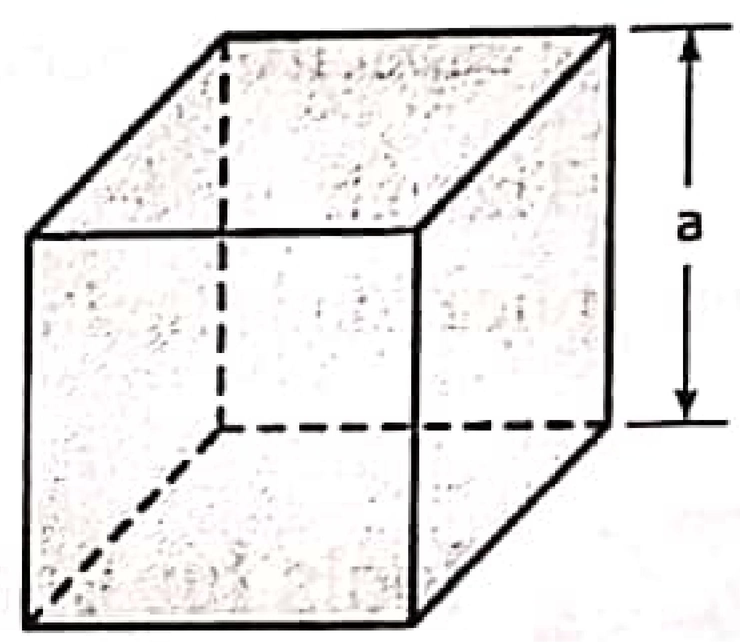


Fig. 5.3 (a)

The surface area of the cube is

$$S = 6a^2$$

The volume of the cube is

$$V = a^3$$

The surface to volume ratio is given by

$$\frac{S}{V} = \frac{6}{a}$$

If a = 2 cm, $\frac{S}{V} = 3\ \text{cm}^{-1}$

Now, lets consider a sphere of radius 'a'.

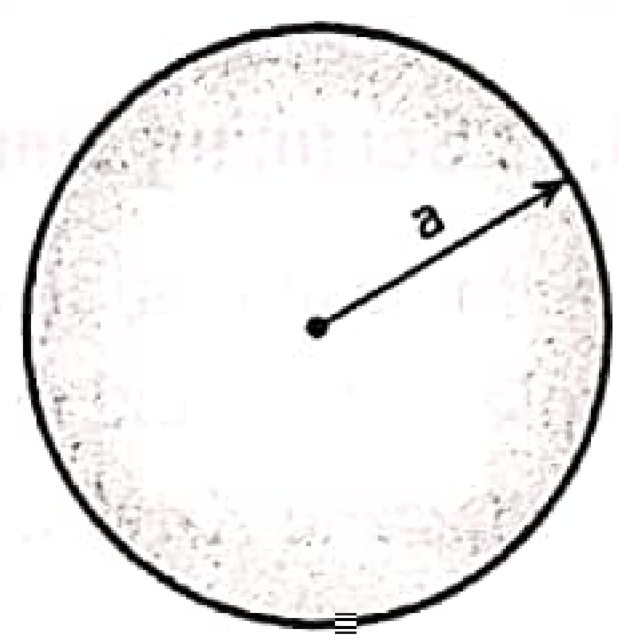


Fig. 5.3 (b)

The surface area is $S = 4\pi a^2$.

The volume is $V = \frac{4}{3}\pi a^3$

The surface to volume ratio is given by

$$\frac{S}{V} = \frac{3}{a}$$

If a = 2 cm, $\frac{S}{V} = 1.5\ \text{cm}^{-1}$

The cube with larger S / V ratio than that of sphere is considered as more efficient in nanotechnology. ***The more the S / V ratio, the greater is the efficiency of the nanomaterial.***

5.3.3 : Two Main Approaches in Nanotechnology

The two approaches used in nanotechnology to prepare nanomaterials are top down approach and bottom up approach which are explained below.

(i) Top Down Approach

In this technique nanostructures are fabricated by reducing a bulk material to nanoparticles through methods as cutting, carving and moulding. Despite the fact that those techniques introduce various structural defects in the material it is widely used in nanotechnology due to its simplicity.

(ii) Bottom Up Approach

In this technique, nanostructures are built up atom by atom or molecule by molecule. Even the nanostructure formed by a single molecule can be developed. The information storage capacity of nanostructures constructed in this approach is very high.

Though this technique does not cause much damage to the structure of the material its application is limited due to the complexities involved.

5.4 Tools for Characterization of Nanoparticles

Several forms of microscopy are available for studying nanomaterials are discussed below. Three most commonly used microscopies are as follows :

5.4.1 : Scanning Electron Microscope (SEM)

In scanning electron microscope an electron beam is made to be incident on the sample surface and its image is formed by the emitted secondary electrons, back scattered electrons and X-rays.

Principles

It is based on the wave nature of electrons and the interactions of high energy electrons with the sample surface.

Construction

A schematic diagram of SEM as shown in Fig. 5.4.

- There is an electron gun comprising of a filament and a cathode which emit a beam of thermionically emitted electrons.
- The electron beam passes through two pairs of condenser lenses C_1 and C_2.
- The condensed electron beam then passes through a scanning coil S.
- Before being incident on the sample the electron beam passes through the objective lens.

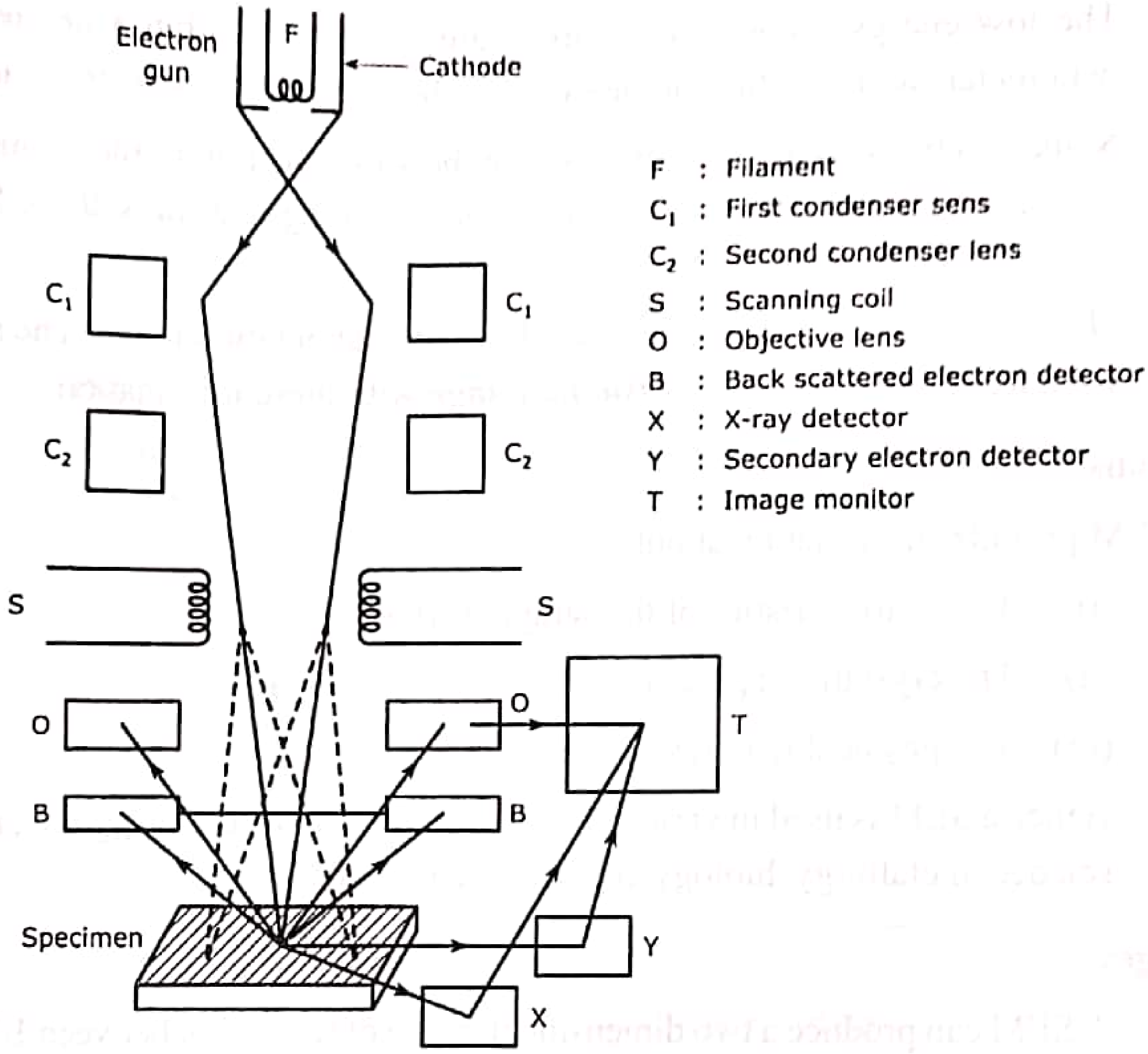


Fig. 5.4 : Scanning Electron Microscope (SEM)

- The detectors are used to detect the back scattered electrons, the secondary electrons and the X-rays.
- Taking input from the detectors the image is produced on the monitor.

Working

- The electron gun produces a high energetic electron beam.
- The condensed lenses focus the diverging electron beam into a fine beam of a spot diameter of few nanometers.
- The scan coils deflect the electron beam in various directions to scan across the surface of the sample.
- The objective lens is used to focus the beam at a particular point on the sample surface.
- The back scattered electrons are reflected from the surface of the sample and are collected by the detector, B.

- The low energy secondary electrons are originated within a depth of few nanometers of the surface of the sample. These are collected by the detector Y.
- Some electrons of the incident electron beam go deep in to the atoms of the sample and knock off inner shell electrons resulting in X-rays, these X - rays are collected by detector X.
- Hence the detectors collect all the information about the sample. The monitor produces the final ***two dimensional image*** with these informations.

Applications

A SEM provides information about

(i) The characteristics of the sample surface.

(ii) The crystallographic structure of the specimen.

(iii) The physical features of the sample.

- Hence a SEM is used in various fields of science and technology *e.g.*, material science, metallurgy, biology, medical science etc.

Advantages

- A SEM can produce a two dimensional image of resolution between 10 A° and 100 A°.
- A SEM has a very high magnifying power.

Disadvantage

- A SEM can produce an image of the surface of the sample and not of its interior.
- The sample to be studied with a SEM is required to be conducting. For no conducting samples a thin conducting coating on the top surface is used.

5.4.2 : Scanning Tunneling Microscope (STM)

A scanning tunneling microscope (STM) is a very powerful microscope which can produce images of individual atoms of the sample surface.

Principle

The STM works on the principle of ***quantum mechanical tunneling effect which means that the de Broglie electron waves tunnel through A thin insulating layer between two conducting materials.***

Construction

- The STM consists of a probe with a very fine tip made up of tungsten or gold. The probe is fitted to a cantilever so that it can move over the surface of the sample. The distance between the tip and the sample surface is maintained around 1 nm. The tip of the probe is so fine that it is possible to scan the sample surface atom by atom.
- The probe is maintained at a positive potential and the sample is maintained at a negative potential with an insulating air gap between then, this is shown in Fig. 5.5.

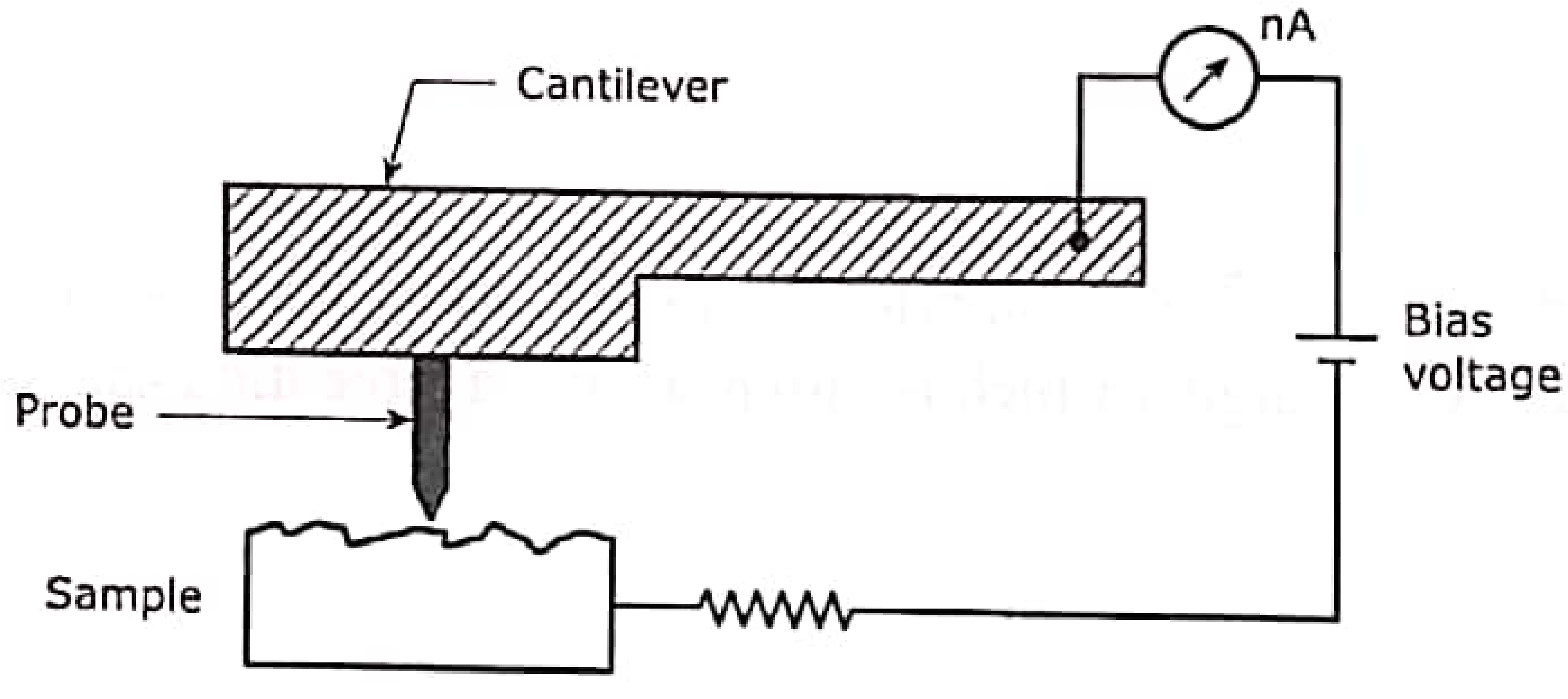


Fig. 5.5 : Scanning Tunnelling Microscope

Working

When the biasing is done electron waves from the tip of the probe tunnel through the air gap and reach the sample surface, giving rise to the probe current.

A STM works on two different modes :

(i) **Constant height mode :** In this mode the tip is moved on the sample surface at a constant height around 4 A^{0} to 7 A^{0}. Due to the irregularities of the sample surface the probe current changes. The probe current is inversely proportional to the thickness of the air gap. The probe current gives the information about each atom of the sample surface with the help of which the image of the surface is produced.

(ii) **Constant current mode :** In this mode, to maintain the probe current constant the height of the probe is varied according to the irregularities of the sample surface. From this height variation of the tip the ***image of the surface topography*** is produced.

In the constant height mode the image information is faster so this is preferable to constant current mode.

Applications

Due to its ability to form ***three dimensional images of the sample*** surface at atomic scale STM has wide applications in the study of characteristic of surfaces, size of molecule. STM is also used to image DNA.

5.4.3 : Atomic Force Microscope (AFM)

The atomic force microscope is a scanning probe microscope used as an imaging device.

Principle

The various types of forces experienced by the probe while scanning the sample surface scatters a LASER signal which is turn produces a three dimensional image of the sample.

Construction

- The AFM consists of a probe with a sharp tip fitted to a cantilever. The radius of the tip is around 1 nm and the length of the cantilever is around 10 nm as shown in Fig. 5.6.

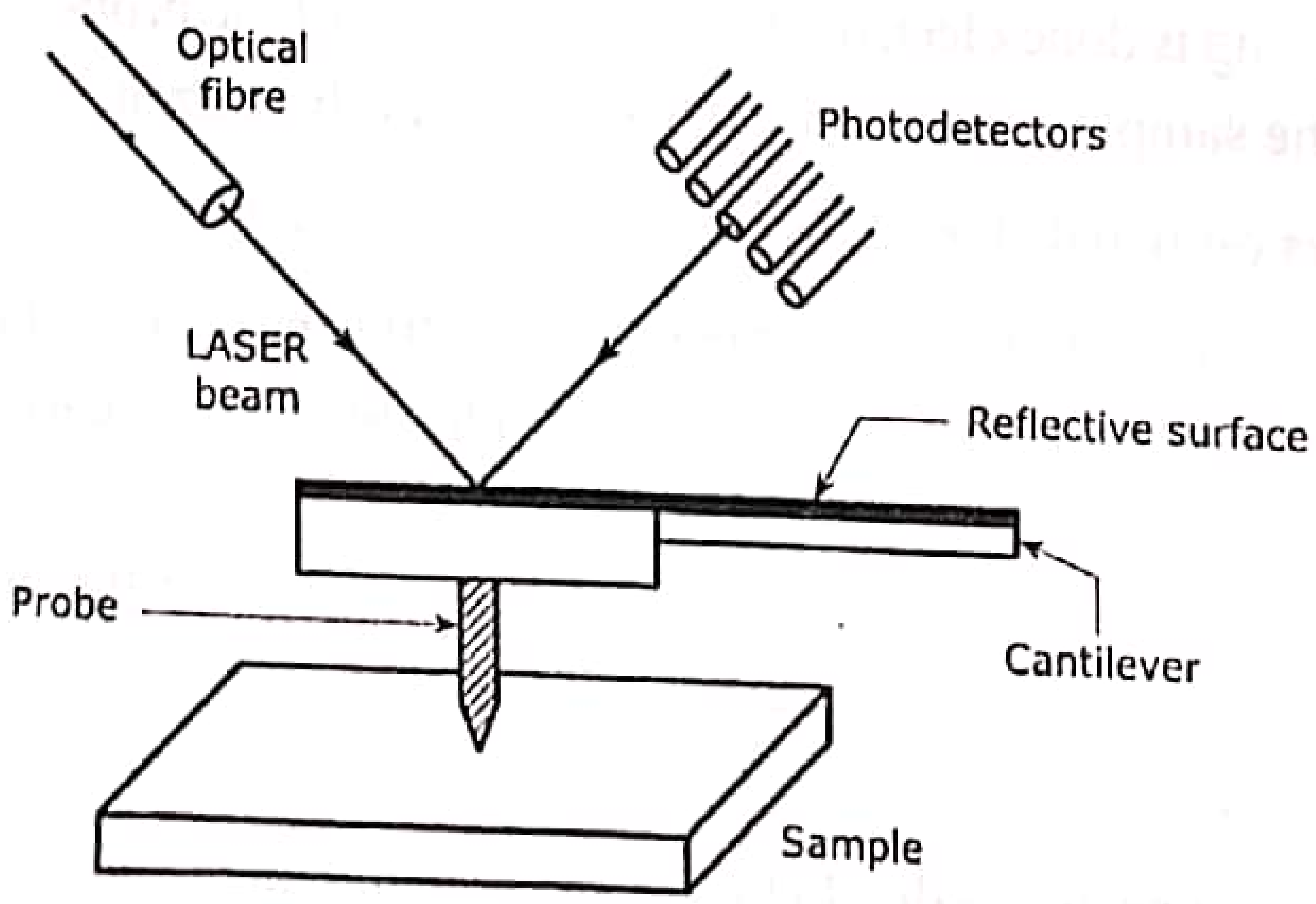


Fig. 5.6 : Atomic Force Microscope

- The cantilever surface is highly reflective. From a LASER source a laser beam and made to be incident on the cantilever through an optical fiber. The reflected LASER beam is collected by a series of photo detectors.

Working

- As the probe is moved over the sample surface the tip experiences a force due to which the cantilever undergoes a deflection.
- According to the type of the sample the force can be of electrostatic, magnetic, mechanical and even van Der Waals forces.
- The interactive force is detected by a series of photo detectors which collect the LASER beam scattered at different direction due to the deflection of the probe.
- The three dimensional image carrying the information of the topography of the surface is then formed.

Advantages

- The resolution of the image is in nanometer range.
- Both the conducting and non conducting surfaces can be scanned by an AFM.

Disadvantage

The scanning process is slow.

Applications

AFM can be used to study various types of samples, *e.g.*, conductors, semiconductors, insulators biological tissues etc. AFM is also used to form nanoparticles with its fine probe.

5.4.4 : Comparison of SEM and AFM

Table 5.1

Sr. No.	SEM	AFM
1.	The sample needs to be conducting.	The sample can be conducting or nonconducting.
2.	The operation requires vacuum.	The operation is possible in open atmosphere.
3.	Resolution of the image is more.	Resolution of the image is less.
4.	It produces a two dimensional image.	It produces a three dimensional image.

5.5 Methods to Synthesize

Various methods used for the production of nanomaterials are described here.

(a) Mechanical Method : Ball Milling Method

In this method small hard steel balls are kept in a container filled with the powder of the bulk material. The container spins about itself while rotating in a circular path about a central axis like a planet moves around the sun. The size of the steel balls used in milling is inversely proportional to the size of the nanoparticles they produce. This is a simple, economical method that can be used at room temperature. This is used to make nanoparticles of metals and alloys.

(b) RF Plasma Technique : Sputtering

- In this technique the bulk material is kept in a pestle which is kept in an evacuated chamber as shown in Fig. 5.7.

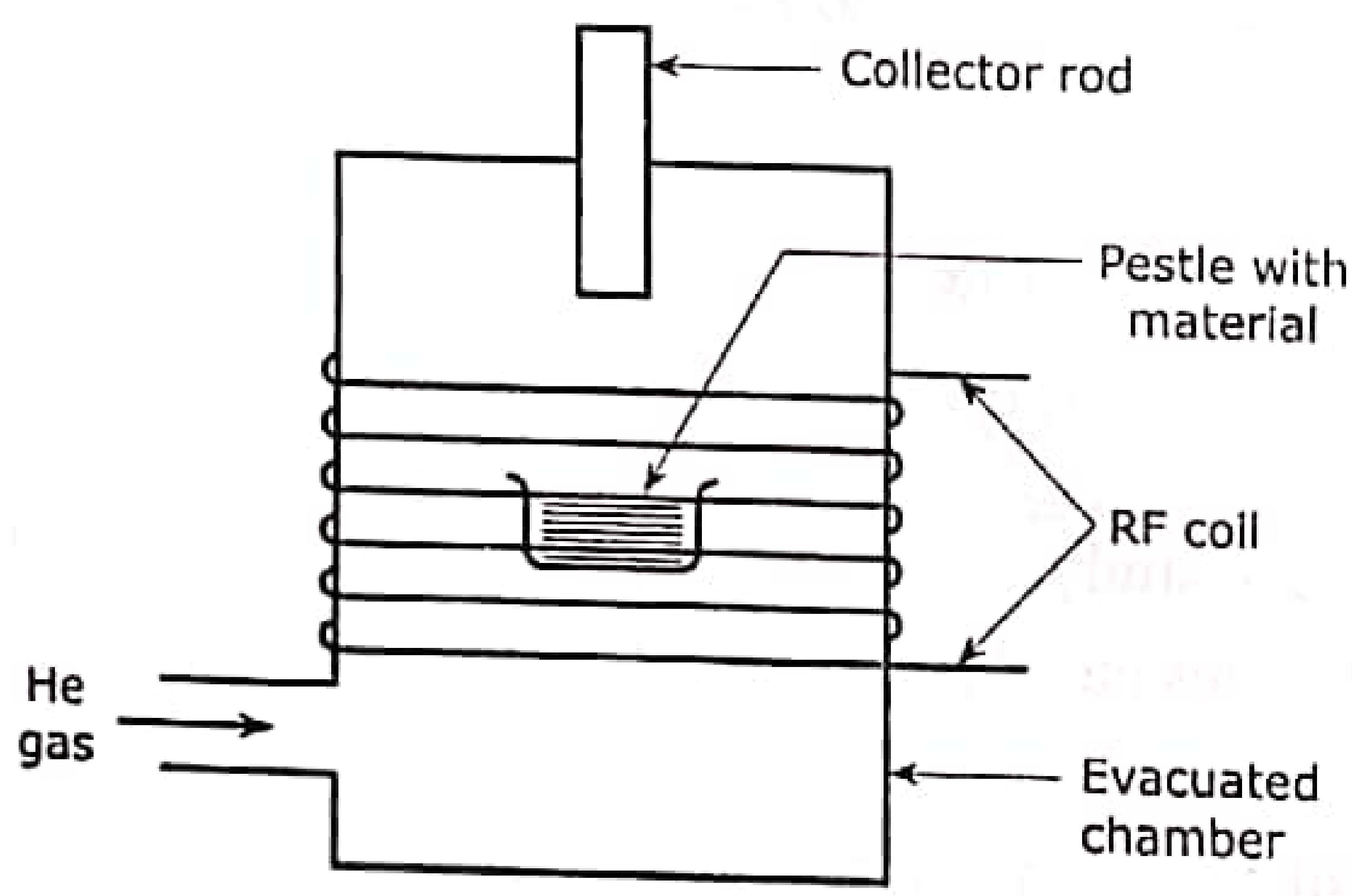


Fig. 5.7 : RF Plasma

- When a high voltage is applied to the RF coils heat in generated and the evaporation of the metal begins. Then cold He gas is allowed to enter the chamber. This results in high temperature plasma in the region of the coils. Nanoparticles are formed from the metal vapor and are collected by the collector.

(c) Inert Gas Condensation : Vapour Diposition

This is the primitive technique of synthesizing nanomaterials. In this technique a metallic or inorganic material is vaporized. In the evaporation process ultrafine particles are formed. These particles rapidly form clusters which in turn condense into crystallites. Using a rotating cylinder and a cold finger both maintained at liquid nitrogen temperature the nanoparticles are removed from the gas. This method is very useful to produce composite materials.

(d) Chemical Solution Deposition Method : Sol-Gel Method

A sol is a solution with particles suspended in it. When the particles in the sol form long polymers throughout the sol it becomes a gel. The sol-gel process is a bottom up approach technique. The bulk material is converted to a powder and mixed in a chemical solution to form the sol. The sol is then partly converted to gel. The sol-gel solution through cavitation effect produce the nanoparticles.

Sol-gel synthesis is superior of all the available processes as it can produce large quantities of nanomaterials at relatively low cost. In this technique almost any material can be synthesized. It is very useful in producing extremely homogeneous alloys and composites controlling the physical, chemical and mechanical properties and the microstructure of the developed nanostructure.

(e) Laser Ablation

In this method a very high intensity (> 10^7 w/cm^2 pulsed laser beam is focused on the material target. The pulsed laser generates very high temperature (> 10^4 K) at the target element resulting in the vaporization of the material. A cool, high-density helium gas is made to flow over the target resulting in the formation of clusters of the target material. The clustered material is then thermalizes to room temperature and finally cooled to a few K to produce nanomaterials.

This technique has an extensive use because of the fact that a wide range of bulk material can be used in this top-down kind of approach.

(f) Thermolysis

In this process the nanoparticles are formed by decomposing solids at high temperature having metal cations and molecular anions or metal organic compounds.

5.6 Applications of Nanomaterials

Nano materials have a wide verity of applications some of which are explained below.

1. **Self cleaning glass :** Nanoparticles are coated on a glass surface to make it photocatalytic and hydrophilic. In photocatalytic effect when UV radiation falls on the glass surface, the nanoparticles become energised and begin to

break down the organic particles on the glass surface. On the otherhand, due to the hydrophilic nature the glass attracts water particles which then clean it.

2. **Clothing :** Clothing with improved UV protection are manufactured by applying a thin layer of zinc oxide nanoparticle on it.

   Also clothes can have nanowhiskers that can make them repel water and other materials thus making them stain resistance.

   Silver nanoparticles coating can have an antibacterial effect on the clothes.

3. **Scratch resistant coating :** Materials like glass are being coated with thin films of hard transparent material to make it scratch resistant.

   Antifog glasses with transparent nanostructures conduct electricity and heat up the glass surface to keep it fog free.

4. **Smart materials :** Nanotechnology enabled smart materials may be able to change and recombine much like the shape shifting cyborg in the movie terminator 2. They may incorporate nonsensors, nanocomputers and nanomachines into their structure which may enable them to respond directly to their environment.

5. **Cutting Tools :** Cutting tools made of nanocrystalline materials are much harder, much wear-resistant and very long lasting.

6. **Insulation Materials :** Nanocrystalline materials synthesized by the sol-gel techniques results in a foam like structure called aerogel. These are porous and extremely light but can withstand heavy weight. These are very good insulators. Aerogels are also used to boost the efficiency of transducers.

7. **Ductile, Machinable ceramics :** Normal ceramics are very hard, brittle and difficult to machine. However, the nanocrystalline ceramics possess good formability, good machinability combined with excellent physical, chemical and mechanical properties.

8. **Low-cost flat-panel electrochromic displays :**

   - Electrochromic devices are very similar to liquid crystal displays. These devices display information by changing color when a voltage is applied. If the polarity of the voltage is reversed the colors gets bleached.
   - If nanocrystalline materials are used in these devices the resolution, the brightness and the contrast of the display increases greatly.

9. **Elimination of pollutants :** Since nanomaterials exhibit enhanced chemical activity they can be used as catalysts to react with pollutants like carbon monoxide and nitrogen oxide to prevent environmental pollution arising from burning gasoline and coal.

10. **High power magnets :** The nanocrystalline magnets have very high magnetic strength given by its coercivity and saturation magnetization value. These magnets have applications in automobile engineering, marine engineering, in medical instruments like MRI etc.

11. **High energy - density batteries :** Nanocrystalline materials synthesized by sol-gel treatment has foam like structure which can store a large amount of energy hence batteries with separator plates made up of these materials do not need frequent changing.

12. **High sensitivity sensors :** Sensors made of nanocrystalline materials are extremely sensitive to the change in their environment. These sensors are used as smoke detectors, ice detectors on aircraft wings, automobile engine performance sensor etc.

13. **Aerospace components :** Aerospace components made of nanomaterials are stronger, tougher and more long lasting than those with conventional materials. This increases the life of the aircraft greatly.

Important Points to Remember

1. **The surface area to volume ratio :** This determines the efficiency of the object.
2. **Two approaches :** Top down and bottom up.
3. **SEM :** Electron wavelength : $\lambda = \frac{h}{\sqrt{2meV}}$.

   Sample needs to be conducting.

   Operation are possible only in vacuum.

   A high resolution two dimensional image is formed.
4. **STM** works on quantum mechanical tunneling effect.

   Sample should be conducting.

   A three dimension contour of the sample surface is imaged at atomic scale.

5. **AFM :** Sample may be conducting or nonconducting.

   A three dimensional image of the topography of the surface is formed.

   Resolution is very high.

6. **Nano materials :** Nano layers - 1 dimension in nano scale

   Nanotubes / nanowires - 2 dimensions in nano scale

   Nanoparticles - 3 dimensions in nano scale

EXERCISE

1. What are nanomaterials and what are their different types?
2. Explain the significance of 'surface area to volume ratio'.
3. Explain top down and bottom up approaches.
4. With schematic diagram explain the principle, construction and working of scanning electron microscope.
5. With schematic diagram explain the principle, construction and working of a scanning tunneling microscope.
6. With schematic diagram explain the principle, construction and working of a Atomic Force microscope.
7. Compare the SEM and AFM.
8. Discuss different methods to synthesize nanomaterials.
9. Discuss various applications of nanomaterials.

Previous University Examination Questions with Solutions

1. Explain top down and bottom up approaches to prepare nanomaterials. Explain one of the methods in detail. **(M.U. Dec. 2015, 16, 17; May 2015, 19) (5 m)**

   [Refer § 5.3.3]

2. What is the difference between bottom up approach and top down approach with respect to nanotechnology? **(M.U. May 2017) (5 m)**

   [Refer § 5.3.3]

3. Draw the schematic diagram of SEM and explain its construction and working.
(M.U. May 2013, 14, 17, 18; Nov. 2018; Dec. 2013, 14, 16, 19) (5 m)
[Refer § 5.4.1]

4. Explain the construction and working of atomic force microscope.
[Refer § 5.4.3] **(M.U. Dec. 2012, 16, 17; May 2015, 16, 19) (5 m)**

5. What are different techniques to synthesize nanomaterials? Explain one of them in detail. **(M.U. May 2013, 17; Dec. 2016, 19) (5 m)**
[Refer § 5.5]

6. Explain the physical methods for synthesis of nanoparticles. **(M.U. May 2014) (5 m)**
[Refer § 5.5]

7. What are nanomaterials? Explain one of the methods of its production in details.
[Refer § 5.5] **(M.U. May 2018) (5 m)**

8. Explain sputtering method for synthesis of nanomaterials. **(M.U. May 2019) (5 m)**
[Refer § 5.5 (b)]

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MODULE 6

Physics of Sensors

(**Prerequisites :** Transducer concept, Meaning of calibration, Piezoelectric effect.)

Resistive Sensors :

(a) Temperature measurement : Pt100 construction, calibration,

(b) Humidity measurement using resistive sensors.

Pressure Sensor : Concept of pressure sensing by capacitive, Flux and inductive method, Analog pressure sensor — Construction, Working and Calibration and Applications.

Piezoelectric Transducers : Concept of piezoelectricity, Use of piezoelectric transducer as ultrasonic generator. Application of ultrasonic transducer for distance measurement, Liquid and air velocity measurement.

Optical sensor : Photodiode, Construction and use of photodiode as ambient light measurement and flux measurement.

Pyroelectric Sensors : Construction and working principle, Application of pyroelectric sensor as bolometer. **(05 Hours) (Weightage - 15%)**

Course Outcome : CO6 : Learner will be able to interpret and explore basic sensing techniques for physical measurements in modern instrumentations.

SYNOPSIS

6.1 Introduction

6.2 Prerequisites

6.3 Resistive Sensors : Resistive Transducers

6.4 Pressure Sensor or Pressure Transducer

6.5 Piezoelectric Transducers

6.6 Optical Sensors

6.7 Pyroelectric Sensors

Important Points to Remember

Exercise

6.1 Introduction

A sensor is a device which converts a physical property into an electrical property (such as resistance). A sensing system is a system, usually a circuit, which allows this electrical property and as the physical property to be measured.

6.2 Prerequisites

6.2.1 : Transducer

A transducer is defined as a device that receives energy from one system and transmits it to another, often in a different form. The energy transmitted by transducers may be electrical, mechanical or acoustical.

6.2.2 : Calibration

Calibration is an essential part of industrial instrument and control. **Calibration can be defined as the comparison of specific values of the input and output of an instrument with a corresponding reference standard.** Though calibration does not guarantee the performance of an instrument, it offers a guarantee to the device or instrument that it operates with the required accuracy and the range specifications under the stipulated environmental conditions. Calibration must be performed periodically to test the validity of the performance of the device or the system.

6.3 Resistive Sensors : Resistive Transducers

Resistive sensors are resistive transducers whose resistance varies with various physical quantities like temperature, pressure, force displacement, vibration, etc.

The resistive transducers convert the physical quantities into variable resistance which is easily measured by the meters.

Principle

The resistive transducer element works on the fact that the resistance (R) of an element is directly proportional to the length (l) and inversely proportional to the area (A) of the conductor, *i.e.*,

$$R = \rho \frac{l}{A} \qquad \text{.....................} (6.1)$$

where ρ is the resistivity of the conductor.

6.3.1 : Temperature Measurement : Pt 100 Sensor

The resistance of a conductor changes when its temperature is changed. This property is utilised for the measurement of the temperature. The resistance thermometer is an instrument used to measure electrical resistance of the conductor to determine the temperature.

The main part of a resistance thermometer is its sensing element. The characteristics of the sensing element determines the sensitivity and the operating temperature range of the instrument.

The sensing element maybe any material that exhibits a relatively large resistance change with change in temperature. Also the material used should be very stable in its characteristics which is necessary to maintain the calibration of the resistance thermometer.

Platinum, Nickel and Copper are the metals most commonly used as the sensing element. Nickel and Copper being less expensive are used in low range industrial applications. Platinum though expensive has high stability and wide operating range (– 260° C to 1100° C) and hence is commonly used for most laboratory work and for industrial measurements of high accuracy.

Pt 100 Sensor

Pt 100 sensors are the most common type of platinum resistance thermometer. Here, Pt is the symbol of Platinum and 100 refers to the fact that at OC the sensor has a resistance of 100 Ohms.

Principle : The relationship between the temperature and resistance of a conductor in the temperature range near 0° C can be calculated from the equation

$$R_t = R_o (1 + \alpha \Delta t) \quad \text{.................... (6.2)}$$

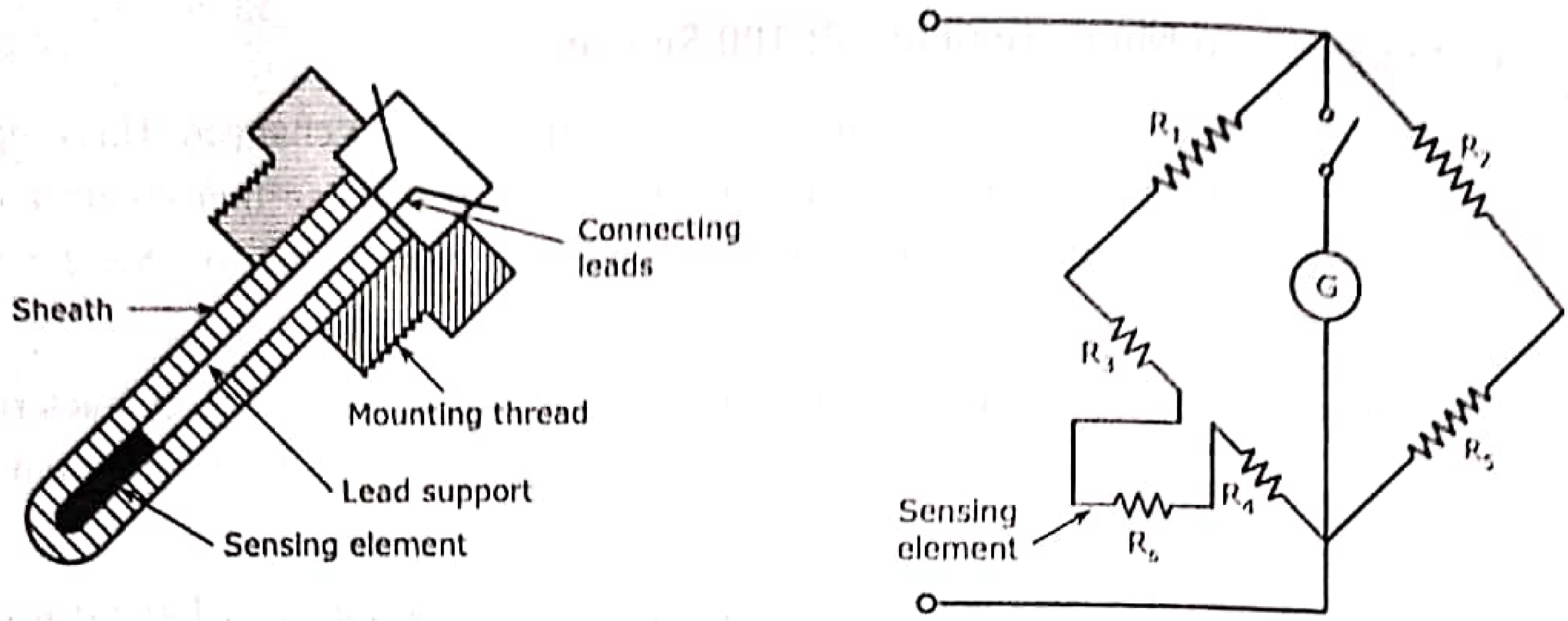
where, R_t = resistance of the conductor at t C.

R_o = resistance of the conductor at the reference temperature (usually C)

Δt = difference between the operating and the reference temperatures

α = temperature coefficient of resistance of the conductor.

Almost all metallic conductors have a positive temperature coefficient of resistance and so their resistance increases with an increase in temperature. A high value of α is desirable in a temperature sensing element so that a substantial change in resistance occurs for a relatively small change in temperature.



(a) Sensing element : Pt 100 thermometer construction (b) Circuit diagram of Pt 100

Fig. 6.1 : Platinum resistance thermometer Pt 100

The construction of an industrial Pt100 sensor is shown in Fig. 6.1 (a). The changes in the resistance of the sensor caused by the changes in temperature are detected by a Wheatstone bridge circuit as shown in Fig. 6.1 (b).

The sensing element R_s is made of a material having a high temperature coefficient, α and R_1, R_2 and R_5 are made of resistances that are practically constant under normal temperature changes. When no current flows through the galvanometer the normal principle of Wheatstone bridge states the ratio of resistance is

$$\frac{R_1}{R_2} = \frac{R_s}{R_5} \qquad \text{..................... (6.3)}$$

When R_s changes due to a change in the temperature the galvanometer shows a deflection and knowing α of the sensing element the temperature can be determined.

6.3.2 : Thermocouple

Thermocouples are temperature sensors that work on thermoelectricity. Thermoelectricity is the electrical energy generated by a temperature difference by thermoelectric effect. The thermoelectric effect is the direct conversion of temperature differences to electric voltage and vice-versa.

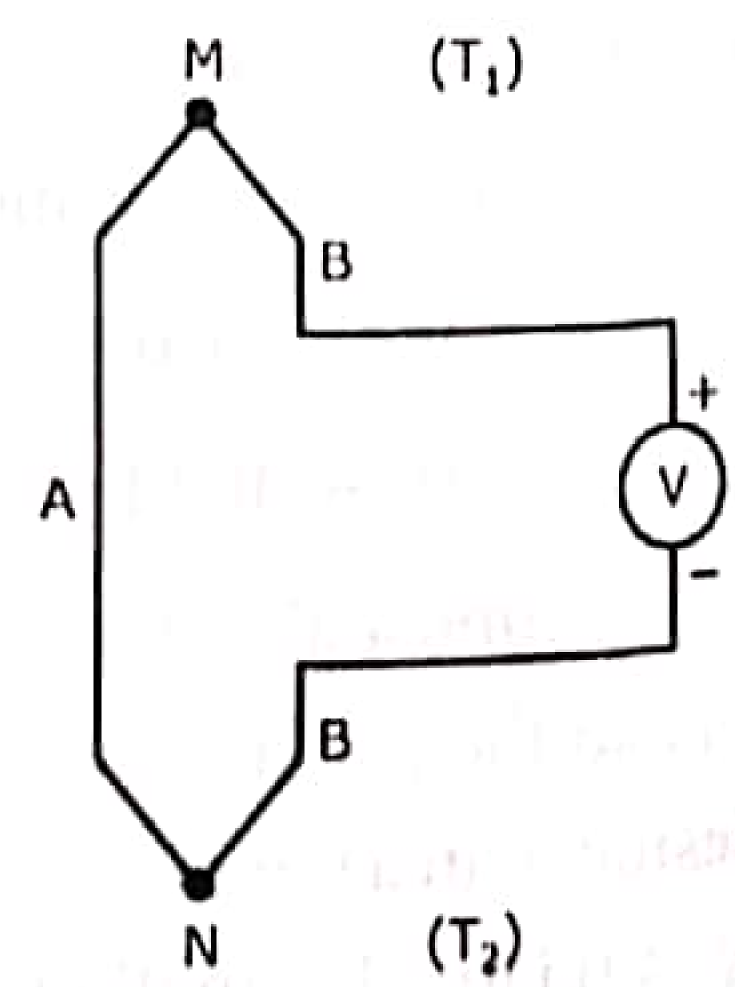


Fig. 6.2 : Thermocouple

Principle : Seebeck Effect

In Fig. 6.2, junction M and N are seen to be formed by two dissimilar metals A and B. If the junction M and N are held at two different temperatures T_1 and T_2 respectively, a thermoelectric emf is developed across the junctions. The thermoelectric emf causes a current through the loop of the junction. This current is known as **thermoelectric current**. This phenomenon is known as **Seebeck effect.**

The thermoelectric emf generated is of the order of several microvolts per kelvin difference.

The voltage developed in the circuit is proportional to the temperature difference $(T_2 - T_1)$ between the two junctions M and N respectively. Hence,

$$V = \alpha (T_2 - T_1)$$

where, $\alpha = \alpha_B - \alpha_A$ with α_A and α_B being the Seebeck coefficients of metals A and B respectively.

Origin of Thermo emf

When a temperature gradient is maintained between the two junctions valence electrons diffuse from the hot side to the cold side leaving behind positive ions on the hot side and generating – ve ions on the cold side. This gives rise to a thermocouple voltage.

The thermo emf generated varies with temperature as

$$e = at + bt^2 \qquad \text{.................... (6.4)}$$

re, a and b are called Seebeck constants of the thermocouple. Equation (6.4) is known ebeck equation.

If the temperature of the cold junction is t at 0° C and the temperature of the hot junction is varied the generated thermo emf exhibits a parabolic behaviour as shown in Fig. 6.3.

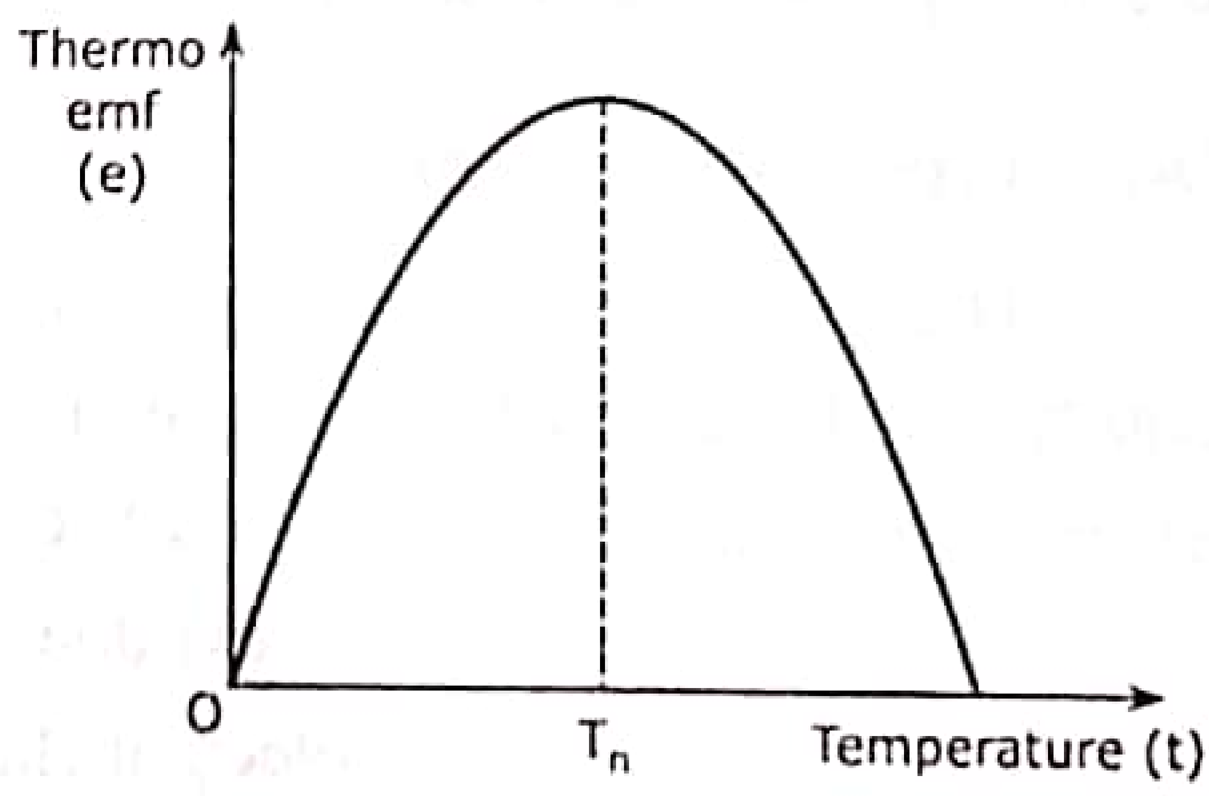


Fig. 6.3

The thermo emf increases parabolically with the temperature and becomes maximum at a temperature T_n, known as the neutral temperature which is constant for a given pair of metals. After this as the temperature of the hot junction increases at a temperature T_i known as the **inversion temperature** the thermo emf becomes zero.

The graph shown in Fig. 6.3 is called the ***calibration curve*** of a given thermocouple.

The slope of the calibration curve is found from equation as

$$\frac{de}{dt} = a + bt$$

At $t = t_n$, $\frac{de}{dt} = 0$

$$\therefore \quad a + b\,t_n = 0 \qquad \therefore \quad t_n = -\frac{a}{b} \qquad \text{.................... (6.5)}$$

At $t = t_i$, $e = 0$

From equation,

$$0 = a\,t_i + b\,t_i^2$$

$$\therefore \quad T_i = -\frac{2a}{b} \qquad \text{.................... (6.6)}$$

This follows from equations (6.5) and (6.6) that

$$T_i = 2\,T_n \qquad \text{.................... (6.7)}$$

To measure an unknown temperature the hot junction is kept at that temperature, the cold junction being maintained at 0° C. The thermo emf generated by the thermocouple is noted and the corresponding temperature from the calibration curve is read. Thus, the unknown temperature is determined.

6.3.3 : Type J and Type K Thermocouples

(A) Type J Thermocouple

The type J is very common and general purpose thermocouple made up of iron and constantan. It has small temperature range and a short life span at high temperatures. It has a sensitivity of approximately 50 μV/C and a narrow temperature range of – 40 C to 750 C. The temperature range is narrow due to the Curie point of iron being 770 C where iron undergoes an irreversible molecular change. The J type thermocouple is a popular one that is commonly used to monitor temperatures of inert materials and in vacuum applications. This thermocouple is succeptible to oxidation. So it is not recommended for damp conditions or low temperature monitoring.

(B) Type K Thermocouple

The type K thermocouple is a commonly used general purpose thermocouple made up of chromel (90% Nickel and 10% chromium) and alumel (95% Nickel, 2% Aluminium, 2% Manganese and 1% Silicon). This thermocouple is inexpensive and reliable with a wide temperature range from – 200 C to 1350 C, high accuracy and a sensitivity of 41 μV/C.

Type K thermocouples are used for measurement in many different types of environments such as water, mild chemical solutions, gases and dry areas. Engines, oil heater and boilers are examples of places where they may be found. These are used as thermometers in hospitals and food industry. The type K thermocouples are commonly used in nuclear applications because of its relative radiation hardness.

6.3.4 : Humidity Measurement Using Resistive Sensors

The device used to measure humidity is called a Psychrometer. This device consists of a thermocouple with one dry bulb and one wet bulb. The thermocouple used in a psychrometer is a chromel - constantan thermocouple.

The dry bulb is simply left exposed to the air to measure the temperature. The wet bulb is covered with a cloth which and dipped in distilled water until it is ready to use. While measuring the humidity the wet bulb is then kept exposed to air. As the water evaporates it cools the wet bulb. By measuring the cooling of the wet bulb the quantity of water evaporated and thence the humidity can be determined. Moist air allows a little evaporation and a small decrease in wet bulb temperature and dry air absorbs lot more moisture, causing too much cooling of the bulb.

Because of the simple set up a psychrometer is a cheap but reliable instrument for humidity measurement.

6.4 Pressure Sensor or Pressure Transducer

A pressure sensor is a device which converts an applied pressure into a measurable electrical signal. These are used for pressure measurement of gases and liquids.

Pressure sensors are used for controlling and monitoring in thousands of everyday applications. Pressure sensors can also be used to indirectly measure other variables such as the speed of a fluid or a gas flow, water level and altitude.

The sensors have a sensing element of constant area and respond to force applied to this area by fluid pressure. The force applied will deflect the diaphragm inside the pressure transducer. The deflection of the internal diaphragm is measured and converted into an electrical output. This allows the pressure to be monitored by microprocessors programmable controllers and computers along with similar electronic instruments.

Some sensors are pressure switches which automatically turns on or off at a particular pressure.

There are two kinds of pressure sensors :

1. **Analog pressure sensors :** These sensors work by converting pressure into an analog electrical signal.
2. **Digital pressure sensors :** These sensors convert pressure into a digital electrical signal.

6.4.1 : Capacitive Pressure Transducer

Capacitive pressure sensors are devices used to measure pressure by detecting the change in the electrical capacitance caused by the movement of a diaphragm due to the pressure applied.

Principle

The capacitive pressure sensor operates on the principle that if the sensing diaphragm between two capacitor plates is deformed by a differential pressure an imbalance of capacitance will occur between itself and the two plates.

The capacitance of a parallel plate capacitor is given by

$$C = k \frac{A \epsilon_0}{d} \text{ (farad)}$$

where, A is the area of each plate and d is the spacing between them. The dielectric constant of the insulator is k and $\epsilon_0 = 9.85 \times 10^{12}$ F/m the permittivity of free space.

Construction and Working

Since, the capacitance is inversely proportional to the spacing of the parallel plates any variation in causes a corresponding variation in the capacitance.

In Fig. 6.4, it is seen that each plate forms a capacitor with the sensing diaphragm which is connected electrically to the metallic body transducer.

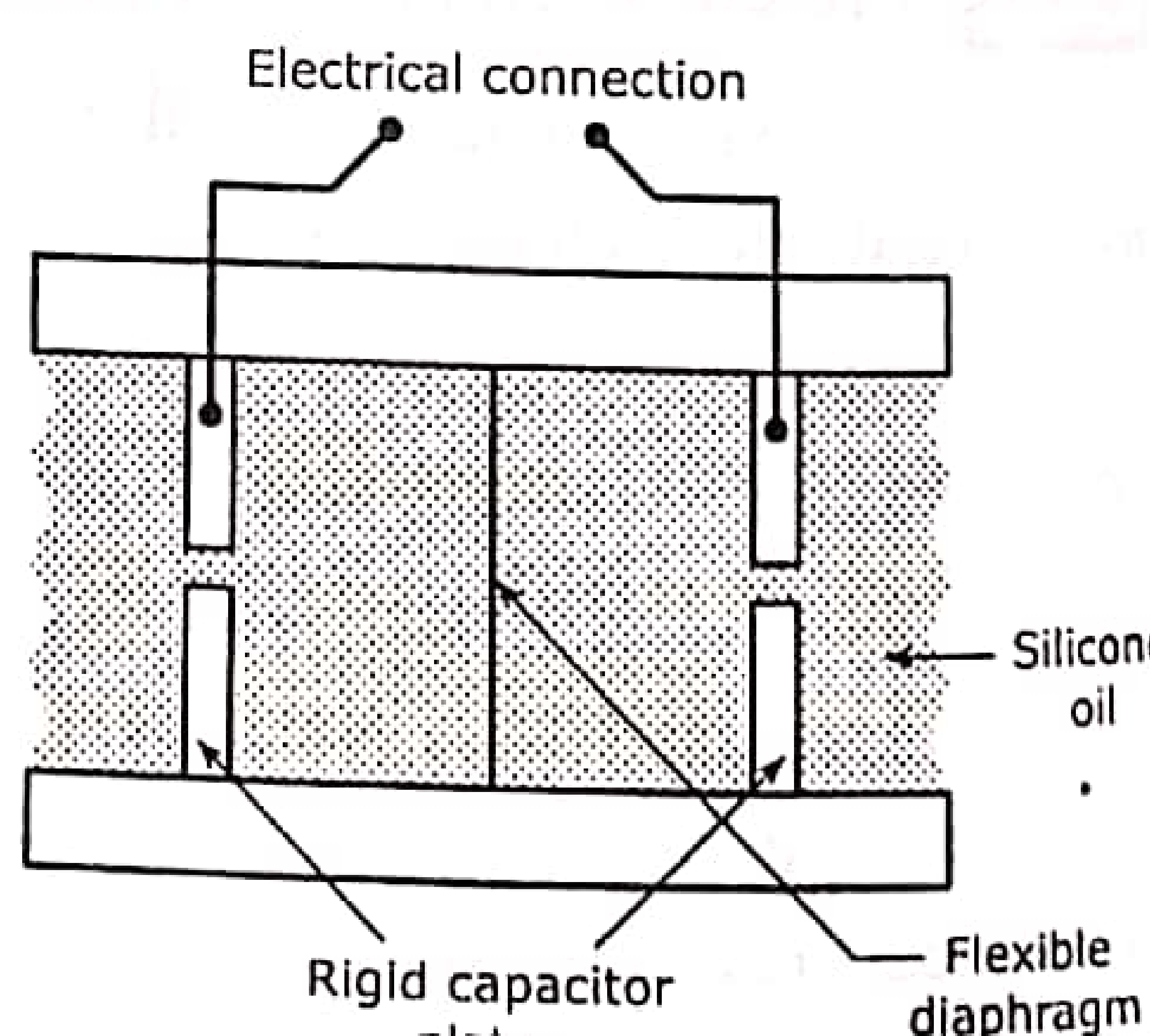


Fig. 6.4 : Capacitive pressure transducer

Two pressures are set up on the diaphragm from its two sides. A net force proportional to these two pressures acts upon the diaphragm and deflects it. The displacement of the diaphragm is sensed as a difference between the capacitances of its two sides.

This change in capacitance is measured using a bridge circuit to measure the equivalent pressure signal.

6.4.2 : Flux and Inductive Method

Inductive sensors are **the devices in which a physical quantity is measured with the help of a change in the self inductance of a single coil or the mutual inductance between two coils.** The physical quantity could be displaement, force, pressure, torque, velocity, acceleration and vibration.

Principle

Inductive sensors work on the principle of magnetic induction of magnetic material, *i.e.*, Faraday's law of induction.

Construction and Working

As seen in Fig. 6.5, the magnetic materials are used in the transducers in the path of the flux. There is some air gap between them. The change in the circuit inductance can be occurred by varying the air gap between them.

An exciter provides a current through the coil and the inductor develops a magnetic field when a current flows through it. Alternately, a current is developed through an inductor when the magnetic field associated with it is varied.

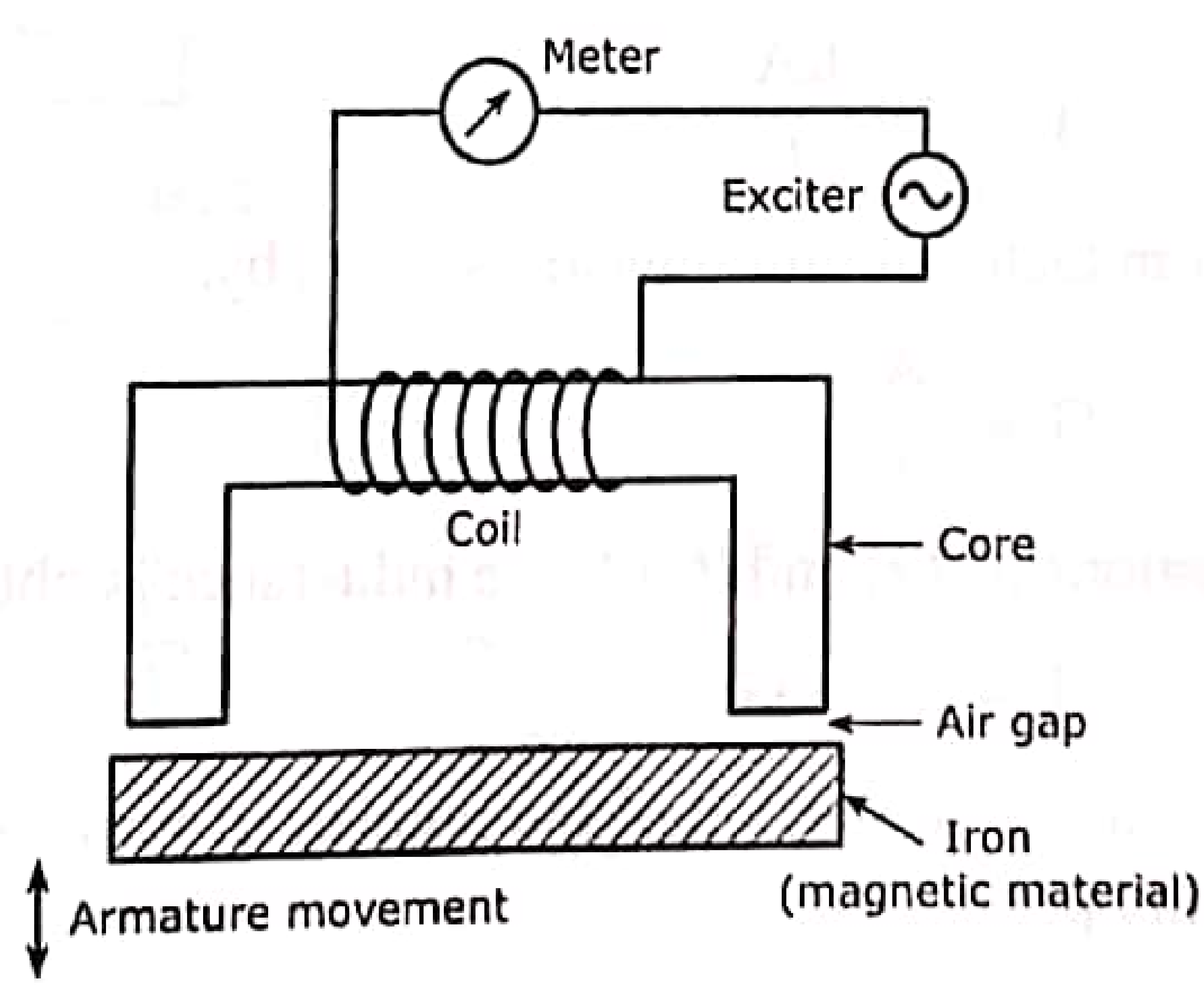


Fig. 6.5 : Inductive transducer working

According to Faraday's law of induction, the temporal variation of the magnetic flux ϕ through a coil of N turns induce a voltage

$$e = -N \frac{d\phi}{dt} \quad \text{.................... (6.8)}$$

When the current i is passed through this inductive coil, the flux produced is

$$\phi = \frac{N i}{R} \quad \text{.................... (6.9)}$$

where R is the reluctance of the coil.

Therefore, substituting equation (6.9) in equation (6.8), we have

$$e = -N \frac{d}{dt}\left(\frac{Ni}{R}\right) = \frac{N^2}{R} \cdot \frac{di}{dt}$$

The self inductance is given by

$$L = \frac{e}{di/dt} = \frac{N^2}{R} \quad \text{.................... (6.10)}$$

Therefore, the output from an inductive transformer can be in the form of either a change in voltage or a change in reluctance.

Also we know that the reluctance R is given by

$$R = \frac{l}{\mu A} \quad \text{.................... (6.11)}$$

where l and A are the length and the cross-sectional area of the coil respectively and μ is the permeability of the medium.

Thus, from equations (6.10) and (6.11), we get the inductance as,

$$I = N^2 \frac{\mu A}{l} \quad \text{.................... (6.12)}$$

The geometric form factor of this inductor is given by,

$$G = \frac{A}{l} \quad \text{.................... (6.13)}$$

Hence, from equations (6.12) and (6.13) the inductance is obtained as,

$$I = N^2 \mu G \quad \text{.................... (6.14)}$$

Hence, the inductance is a function of the number of turns N, the geometric form factor G and permeability μ.

Applications

Inductive transformers are mostly used in the determination of the position and dynamic motion of a metallic object without touching them.

6.5 Piezoelectric Sensors or Transducers

A piezoelectric sensor is an electroacoustic transducer used to measure a pressure or a mechanical stress by converting them into an electrical voltage.

The piezoelectric transducers use special type of materials which induce a voltage when a pressure or stress is applied to it. Such materials are known as electro-resistive materials. Examples of these materials are Quartz, Rochelle salt and Barium Titanate.

6.5.1 : Piezoelectricity

Piezoelectricity is the electricity generated by some solid materials in response to applied mechanical stress.

6.5.2 : Principle of Piezoelectric Sensors : Piezoelectric Effect

Some asymmetric crystalline solids like quartz, tourmaline and Barium titanate exhibit a very special characteristic. Thin slices of these crystals develop a potential difference across the two opposite faces when subjected to mechanical stress in a perpendicular direction as shown in Fig. 6.6. This is known as direct piezoelectric effect. If the direction of the mechanical stress is reversed the potential difference changes its polarity as shown in Fig. 6.6.

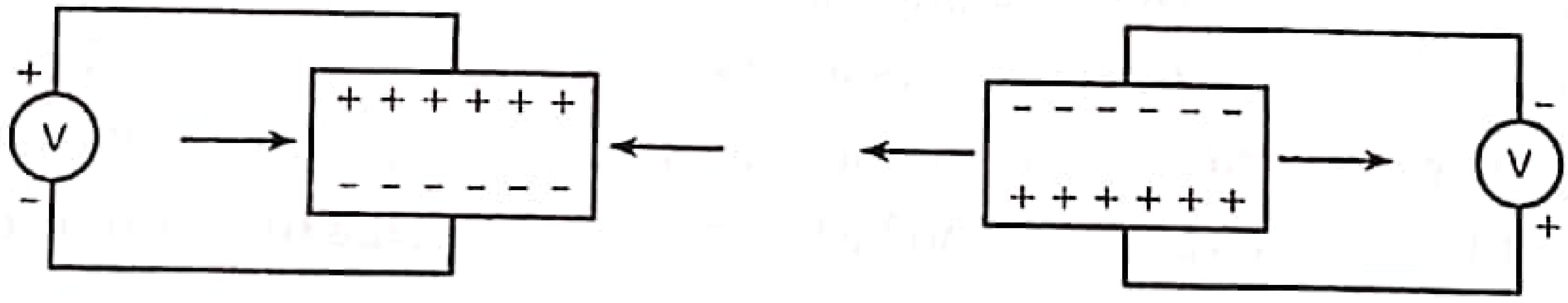


Fig. 6.6 : Direct piezoelectric effect

However, ultrasonic waves are produced by inverse piezoelectric effect which is explained below.

If a voltage is applied across a pair of opposite faces of a piezoelectric crystal it experiences a mechanical stress, *e.g.*, expansion or contraction across the perpendicular faces. If the polarity of the applied voltage is reversed the nature of the mechanical stress is also reversed, *e.g.*, expansion changes to contradiction and contraction changes to expansion as shown in Fig. 6.7.

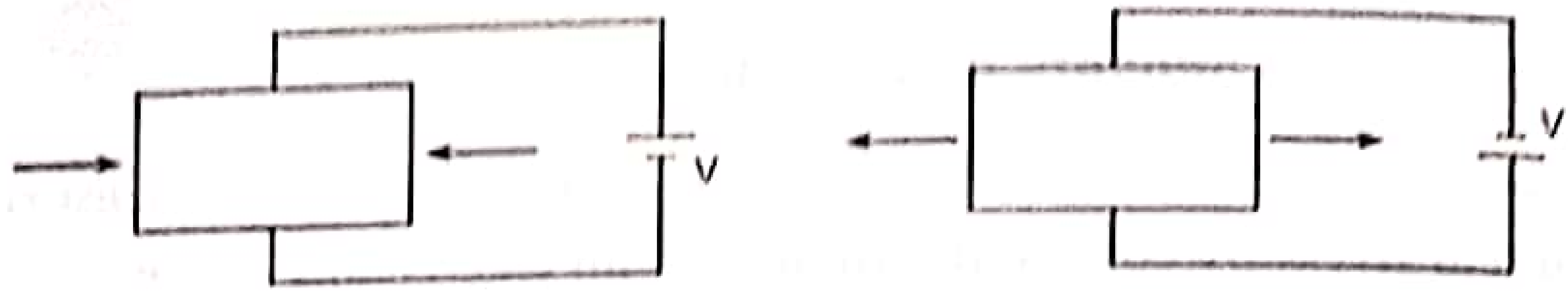


Fig. 6.7 : Inverse piezoelectric effect

When the voltage is changed to an alternating voltage, the crystal slice exhibits alternate extension and contraction and starts vibrating at the frequency of the voltage. If the voltage frequency is increased to the ultrasonic frequency range, *i.e.*, above 20 kHz the crystal slice vibrates at that frequency and ultrasonic of the same frequency is generated.

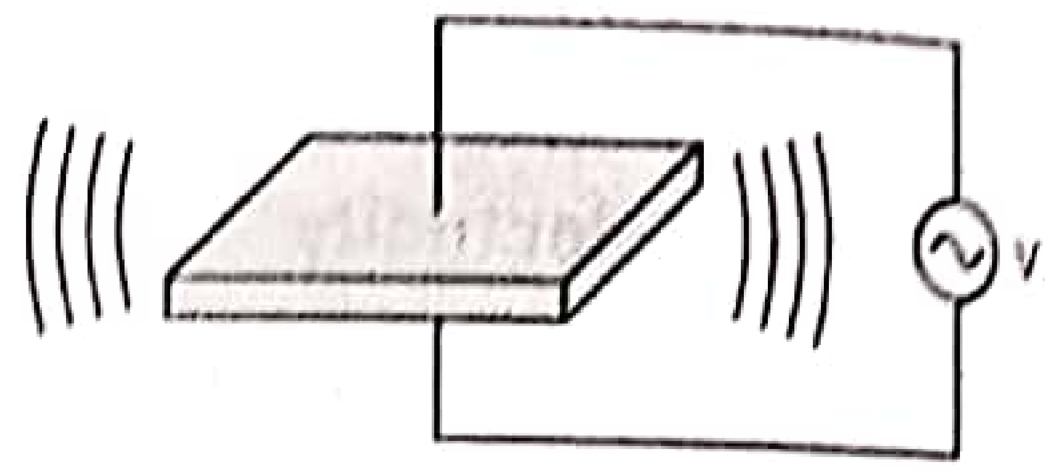


Fig. 6.8

6.5.3 : Piezoelectric Ultrasonic Generator

The circuit diagram as shown in Fig. 6.9 shows that the piezoelectric ultrasonic generator uses a transistor which is biased through a network of resistances R_1, R_2, R_3 and R_4. The coils L_1, L_2 and capacitor C_4 constitute the tuning (resonant) circuit. The tuning circuit is coupled to the transistor T through the coupling capacitor C_2. Capacitor C_3 provides the positive feedback to the amplifier T. The oscillators generated by the tank circuit are sustained and the electrical signal obtained at the output is applied to the electrodes of the piezoelectric crystal through the coupling capacitor C_5. Because of high frequency electrical signal applied to it the piezoelectric crystal produces ultrasonic waves. The frequency of these ultrasonic waves can be varied by the values of the components of the tuning circuit. Ultrasonics of frequency value upto 500 MHz can be generated by this transducer.

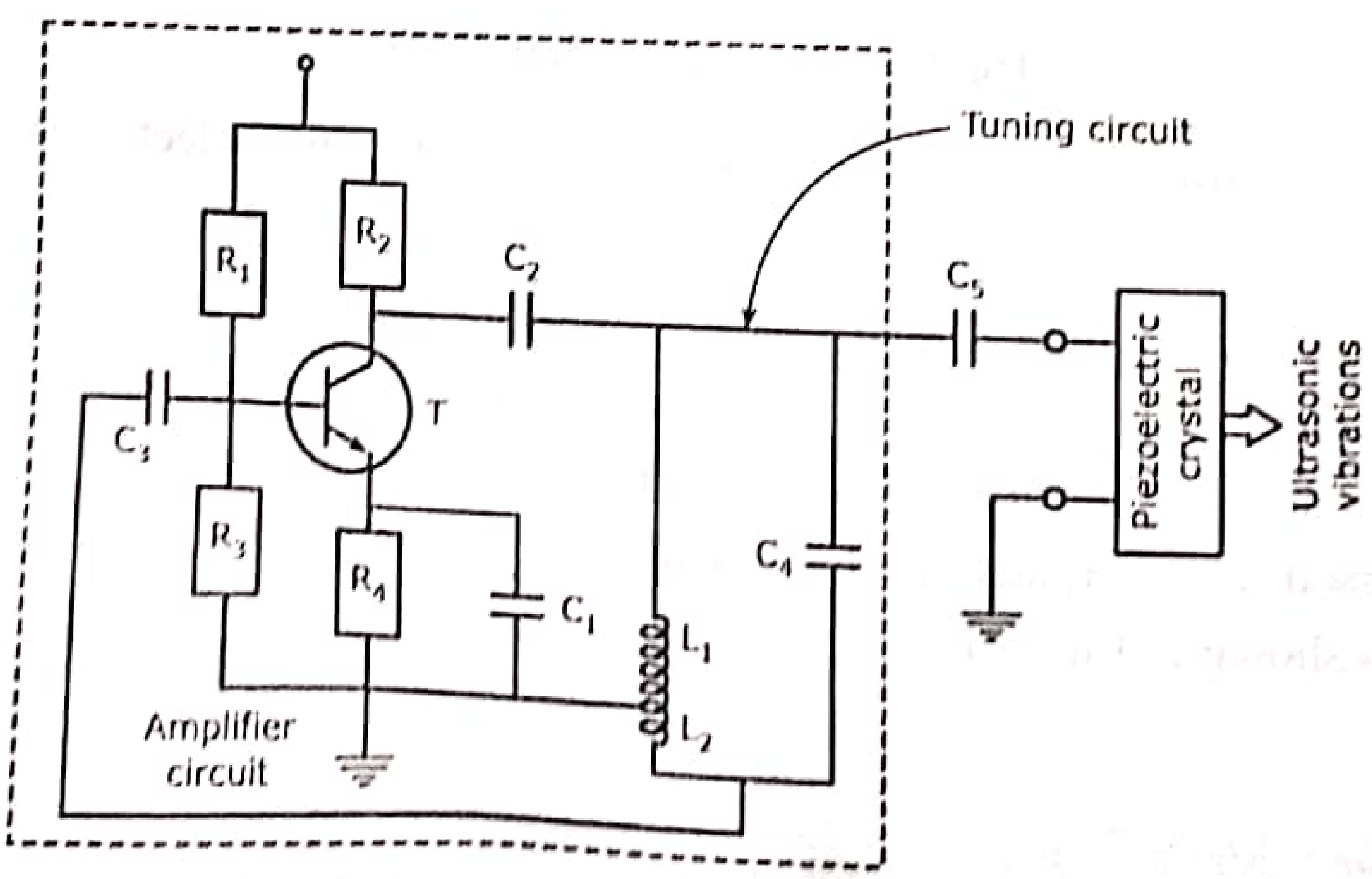


Fig. 6.9 : Piezoelectric ultrasonic generator

6.5.4 : Applications of Ultrasonic Transducers

1. Distance measurement : The method of distance measurement using ultrasonics is based on the pulse-echo method. The ultrasonic sensor emits a high frequency (> 20 kHz) sound pulse. The time taken by the signal to reach the object and travel back to the source after being reflected by the object, is measured. This is called the time of flight. The speed of sound is 341 m/sec in air. Using the following mathematical equation

$$\text{Distance} = \frac{\text{Time} \times \text{Speed of sound}}{2}$$

the distance of the object from the transducer is measured as seen in Fig. 6.10.

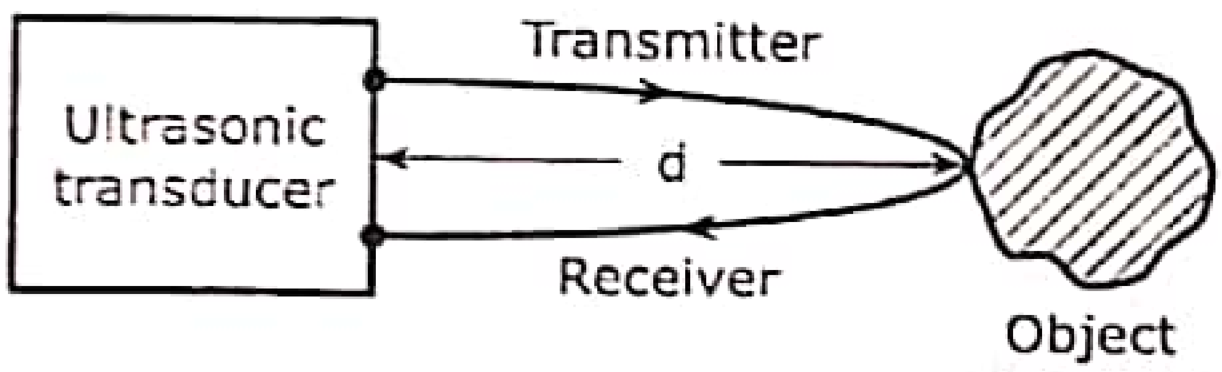


Fig. 6.10 : Ultrasonic distance meter

2. Liquid and air velocity measurement : The method of velocity measurement using ultrasonic waves is based on the pulse-Doppler method. In the case of liquids and gases, the object keeps moving which introduces Doppler effect. Hence, the reflected echo is Doppler shifted as seen in Fig. 6.11.

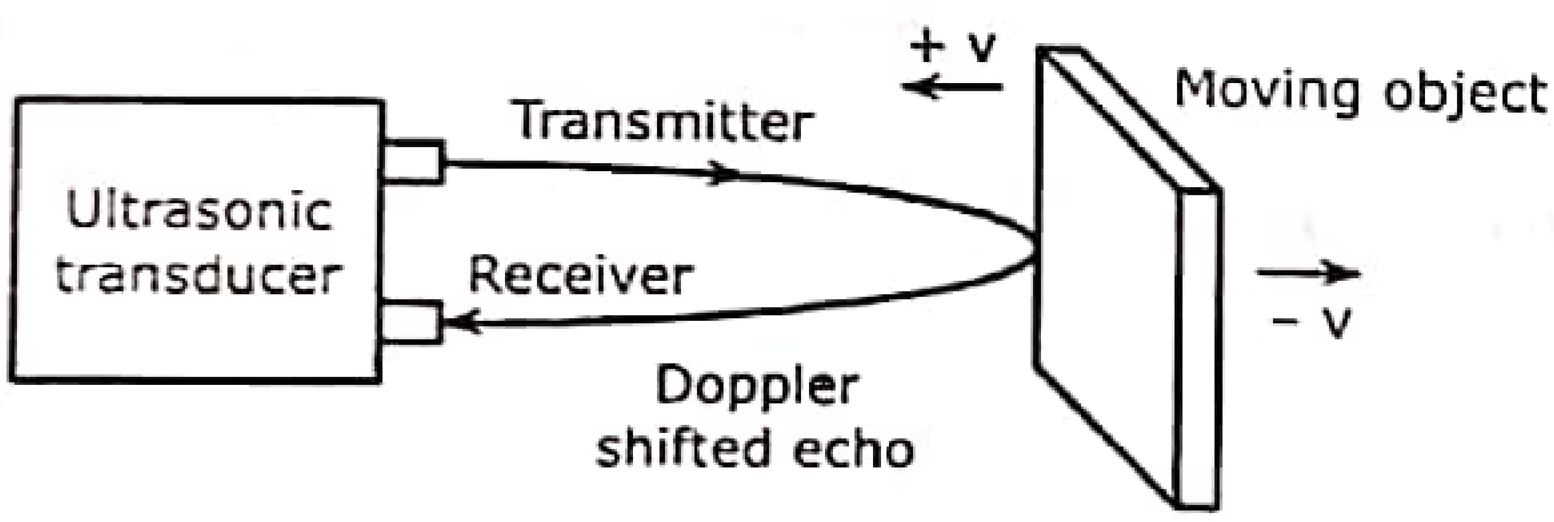


Fig. 6.11 : Velocity measurement by pulse Doppler method

The time of flight in the Doppler shifted echo is proportional to the velocity (v) of the object. Velocity measurement with high resolution requires the calibration of the Doppler shifted time of flight and an accurate distance measurement of the object.

6.6 Optical Sensors

Optical sensors are the transducers that convert optical energy into an electronic signal. The purpose of an optical sensor is to measure a physical quantity of light. Optical Sensors are used for contactless detections.

There are different kinds of optical sensors, the most common types which we have been using in our real world applications are —

(i) Photoconductive devices

(ii) Photovoltaic cells

(iii) Photodiodes.

6.6.1 : Photodiodes

A photodiode is a semiconductor device that converts light into an electric current. The current is generated when photons are absorbed in the photodiode.

Principle : Photoconductivity

Photoconductivity is a phenomenon in which the conductivity of a material increases when it is terminated by radiation.

When radiation with photon energy $E = h\upsilon \geq E_g$ being the band gap energy of the diode material, is incident on a material like semiconductor, it is absorbed by the material. The energy generates electron-hole pairs.

Each photon of energy $h\upsilon$ creates one electron hole pair. The density of the generated electron-hole pair depends on the intensity of the incident radiation. In this way the number of free charge carriers increases which increases the conductivity of the semiconductor.

Construction and Working

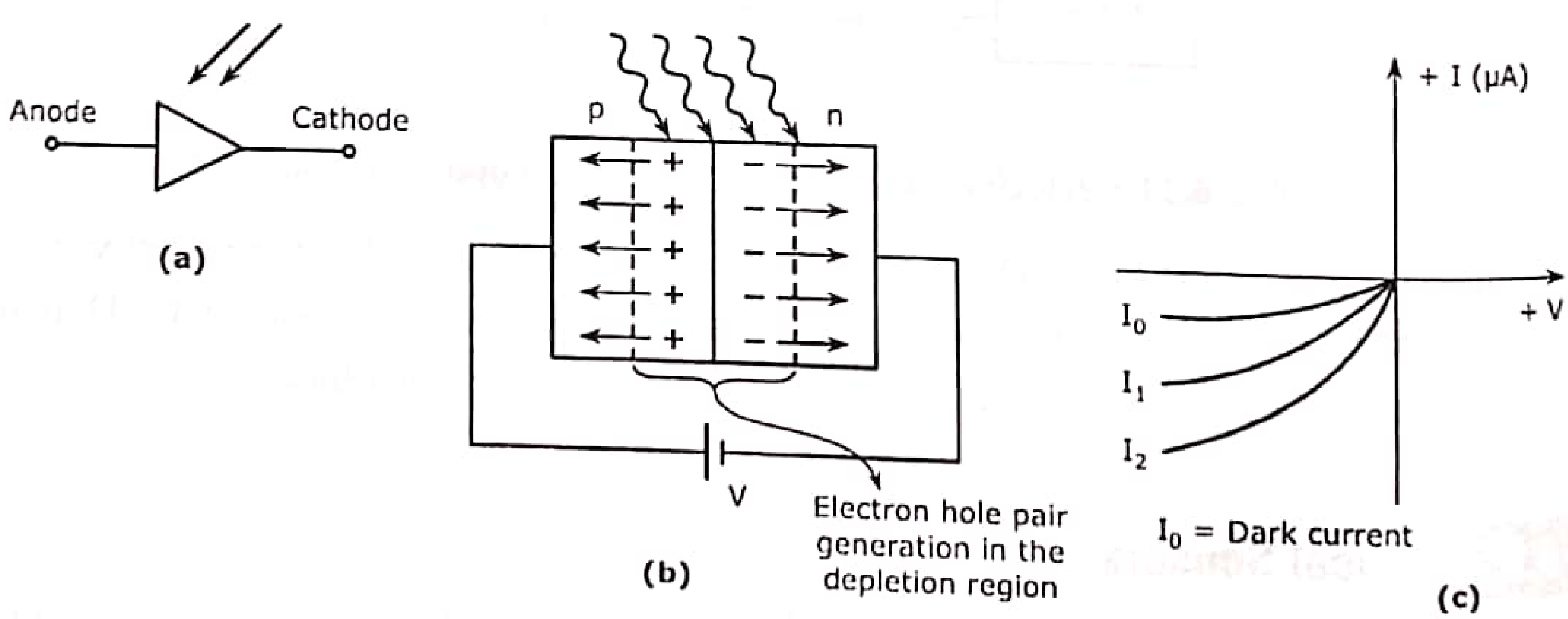


Fig. 6.12 : Photodiodes — (a) Symbol, (b) Circuit, (c) I-V characteristic curve

A photodiode is a normal reverse biased p-n junction diode. Materials commonly used for photodiodes are Si, Ge, PbS and InGaAs. Figure 6.12 (a) shows the symbol of a photodiode. When light is incident on the diode surface as seen in Fig. 6.12 (b) additional electron hole pairs are generated and boost the conduction resulting in an increase in the reverse current. Thus, by controlling the illumination level the current flowing through the device can be regulated. As shown in Fig. 6.12 (c) in reverse biase even in the absence of the incident light, there is a small reverse current due to the minority carriers. This is called dark current.

Applications

1. Ambient light sensors : An ambient light sensor is a component used in smart-phones, electronic notebook, other mobile devices, automotive displays and LCD TVs. It is a photodetector that is used to sense the amount of ambient light present and appropriately dim the screen light of the device to match it. This avoids having the screen be too bright in a dark space or too dim in daylight. Dimming the screen on a mobile device also prolongs the lifetime of the battery.

2. Radiation flux measurement : Radiation flux or radiant power is the radiant energy emitted, reflected, transmitted or received per unit time.

The radiation flus is given by

$$\phi = \frac{L}{4\pi r^2} \quad (W/m^2)$$

where, L is the Luminosity or the total power output of the source and r is the distance from the radiation source.

3. Optical mouse : A very familiar optical sensor is a computer mouse which is an optical mouse. This sensor uses a LED sensor and light detector such as an array of photodiodes to detect movements relative to a surface *i.e.*, the computer screen, in this case.

6.7 Pyroelectric Sensors

Pyroelectric sensors are a kind of thermal sensors in which temperature variations are converted into electrical signals.

Principle

Pyroelectricity : Pyroelectricity is a property of certain crystalline materials called pyroelectric materials. A pyroelectric materials are naturally polarized and as a result contain large electric fields. They also possess a spontaneous polarization when subjected to temperature variations. The change in temperature modifies the position of atoms slightly within the crystal structure such that the polarization of the material changes. This change in polarization gives rise to a voltage across the crystal. If the temperature stays constant at its new value the pyroelectric voltage gradually disappears due to a leakage current.

The most important pyroelectric materials are ceramics, synthetic polymers and ceramic-polymer composites. All pyroelectric materials are also piezoelectric. However, all piezoelectric materials are not pyroelectric. Some piezoelectric materials with crystal symmetry do not allow pyroelectricity. Pyroelectricity is also found to be present in bones and tendons of human body.

Construction and Working

Pyroelectric sensors are constructed from single pyroelectric crystals which are dielectrics. When an electric field is applied across any dielectric material electrical polarization takes place. The magnitude of the polarization is a function of the dielectric constant of the material.

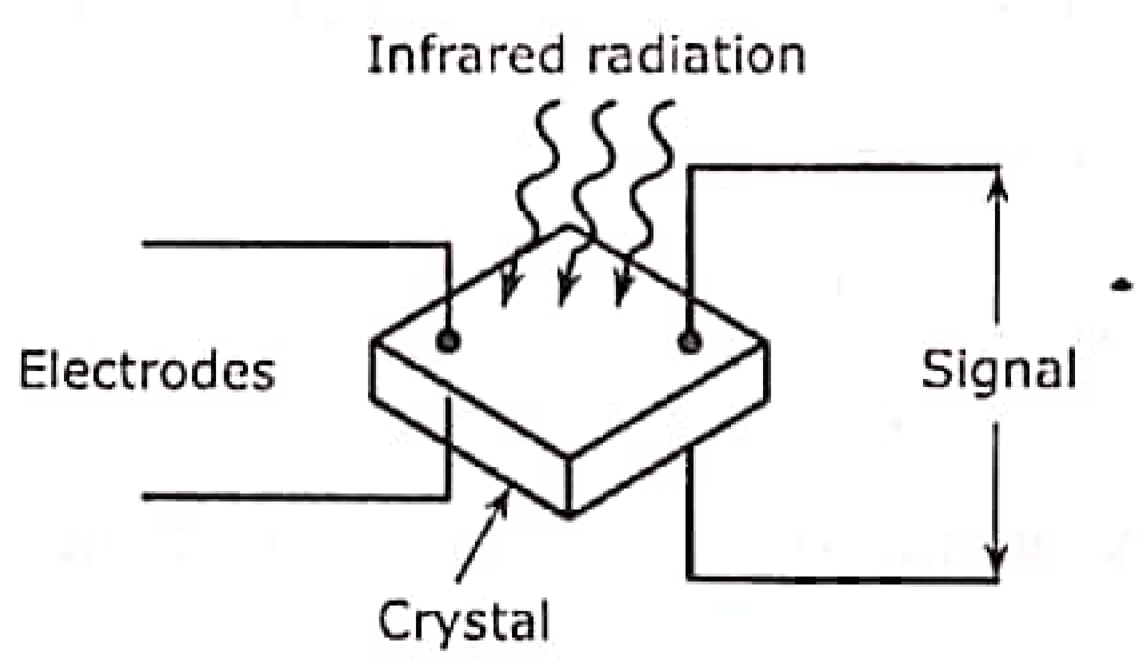


Fig. 6.13 : Pyroelectric sensor

As seen in Fig. 6.13, the pyroelectric crystal is sandwiched between two electrodes which are infrared transparent. When radiation falls on the crystal changes are accumulated on its surfaces and the crystal acts as a temperature dependent capacitor with a resulting voltage $V = Q/C$ where Q is the accumulated charge and C is the crystal capacitance.

As the incident infrared radiation alters the surface charge distribution across the crystal, a current is detected in the external circuit connected to it. The magnitude of this current is directly proportional to the surface area of the crystal and the rate of change of polarization with temperature.

Applications of Pyroelectric sensors

This type of sensors are widely used in —

(i) absolute radiometry

(ii) infrared spectroscopy

(iii) laser interferometry

(iv) simple thermal imaging systems

Important Points to Remember

1. A transducer is a device that receives a kind of energy from one system and transmits it to another device in a different firm.

2. Resistive sensors are resistive transducers whose resistance varies with different physical quantities like temperature, pressure, force, displacement, vibration, etc.

3. Pt100 sensors are made up of platinum (Pt) with a resistance of 100 Ohms at 0° C.

4. Thermocouples are temperature sensors that work on thermoelectricity, *i.e.*, the electricity generated by a temperature difference.

5. A pressure sensor is a device which converts an applied pressure into a measurable electrical signal, used to measure pressure of gases and liquids.

6. Capacitive pressure sensors are devices used to measure pressure by detecting the changes in the electrical capacitance caused by the movement of a diaphragm due to the pressure applied.

7. Inductive transducers are the sensors in which with the help of a change in the self inductance of a single coil or the mutual inductance between two coils a physical quantity is measured.

8. Piezoelectric sensor is an electroacoustic transducer used to measure a physical quantity with the conversion of pressure or mechanical stress into an electrical.

9. Optical sensors are the transducers that convert optical energy into an electronic signal. Optical sensors are used for contactless detections.

10. Photodiode is a semiconductor device that converts light into an electric current.

11. Pyroelectric sensors are a kind of thermal sensors in which temperature variation are converted into electrical signals.

EXERCISE

Short Answer Type

1. Define a transducer. State different types of transducers.
2. What are resistive sensors? State its working principle.
3. Explain the theory of temperature measurement by a resistive sensor.
4. Explain the working principle of a thermocouple.
5. State and explain Seebeck effect.
6. Write short notes on J and K thermocouples.
7. Explain the method of humidity measurement using resistive sensors.
8. Define a pressure sensor.
9. What is piezoelectricity? Explain piezoelectric effect.
10. Explain the concept of distance measurement with an ultrasonic transducer.
11. Explain the velocity measurement by ultrasonic transducer.
12. State and explain two applications of optical sensors.
13. Explain pyroelectricity.
14. State the applications of pyroelectric sensors.

Long Answer Type

1. With a neat diagram, explain the theory and working of a platinum resistance (Pt 100) transducer.
2. Define a pressure sensor. With a neat diagram explain the working of a capacitive pressure sensor.
3. With a neat diagram explain how pressure can be measured by inductive method.
4. Explain piezoelectric effect. With a neat diagram describe ultrasonic generation by piezoelectric method.
5. What is an optical sensor? With a neat diagram explain the construction and working of a photodiode.
6. What are pyroelectric sensors? With a heat diagram explain the principle and working of a pyroelectric sensor.

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